A Novel Numerical Method for Evaluation of Hypersingular Integrals in Electromagnetics

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Abstract

In this study we develop a numerical method for evaluation of hypersingular surface integrals, which arise in the solution of electric field integral equation (EFIE) via Nyström method. Due to the divergent contribution of an infinitesimal area around the singular point, hypersingular integrals are told to be numerically intractable and analytical methods are employed for evaluation of these integrals. In this study we interpret hypersingular integrals as the second order derivative of weakly singular integrals, which can be efficiently evaluated using quadrature rules. By evaluating the derivative of weakly singular integrals numerically, we have shown that hypersingular integrals can accurately be evaluated using the proposed method. We have solved a scattering problem via Nyström method to confirm the validity of the method.

1. Introduction

Electric field integral equation (EFIE) set up an equation which relates the electric field to the unknown current density by using the boundary conditions on the surface or within volume. The main advantage of the method over differential equation methods is that, it only discretizes the domain where sources exist and thereby require lesser number of unknowns [1]. On the other hand, the integrals to be evaluated are singular and complex source-field relations should be evaluated accurately to obtain satisfactory results for the corresponding problem.

A popular method to solve EFIE is the method of moments (MoM) procedure. The method introduces a set of basis functions for the unknown current density and tries to find the coefficients of the basis functions by minimizing projection of error to the space spanned by some testing functions. The basis and testing function are generally selected as divergence conforming which help to reduce the order of singularity of the integral to be evaluated. On the other hand in the locally corrected Nyström (LCN) method, which is introduced in [2, 3] as an alternative to conventional MoM procedure, the integrals are replaced by quadrature rules for faster precompuation and memory reduction. The unknowns in LCN method are samples of current at selected quadrature nodes and therefore no basis and testing functions are used for the unknown current density. In the absence of the divergence conforming basis and testing functions the singular kernel of EFIE should be evaluated directly without reducing the order of singularity. The kernel of EFIE possesses $1/R$ terms and $1/R^3$ terms. The former terms result from the integral of free space Green's function and the latter terms result from the double gradient of free space Green's function. Using the convention of boundary element method we call the surface integral of $1/R$ terms as weakly singular integrals and the surface integral of $1/R^3$ as hypersingular integrals.

The literature for evaluation of weakly singular is diverse and a list of references for analytical or numerical methods can be found in [4]. However the literature addressing evaluation of hypersingular integrals is relatively narrow and few studies introduce methods for these integrals to be used in electromagnetic scattering problems. A Cauchy principal value like approach in the limiting sense is used in [5] and a similar approach along with Stoke's theorem is used in [6] to obtain simpler formulas. Hypersingular surface integrals are converted to regular line integrals on curvilinear patches in [7] but explicit formulas are not introduced. Evaluation of hypersingular integrals on non-planar surfaces is introduced in [8], where Hadamard finite part interpretation is used to eliminate divergent terms. Apart from the above mentioned analytical methods in electromagnetics literature, also numerical methods are developed in mathematical literature [9-11] for evaluation of hypersingular integrals. In the present discussion we interpret hypersingular integrals as the second order derivative of the weakly singular integrals. Unlike the existing numerical methods, in which the singularity of kernel is of $1/R^3$ type, efficient and machine precision methods can be exploited to evaluate weakly singular integrals whose singularity is of $1/R$ type. In this study we utilized double exponential formulas introduced by [12] to evaluate weakly singular integrals numerically. It has been shown in [4] that use of double exponential formulas for numerical integration is far more accurate when compared to the use of Gauss-Legendre quadrature rules. Here we have shown that second order derivative of weakly singular integrals represent hypersingular integrals. Therefore numerical derivation of weakly singular integrals can achieve high accuracy despite the accuracy degradation due to numerical derivation. The method is validated by evaluation of some hypersingular integrals numerically and comparing the result with those obtained using analytical methods. Moreover we have solved a TE^z scattering problem from a perfect electrically conducting (PEC) cylinder and showed that the result is consistent with analytical results.

2. Formulation

In this study we deal with evaluation of the hypersingular integral,

$$\bar{\bar{I}}^h(x_1,x_2) = \vec{\nabla}\iint_{S'} \vec{\nabla}' \frac{1}{\sqrt{(x_1-x_1')^2+(x_2-x_2')^2}} dx_1' dx_2' \qquad (1)$$

where S' is the flat surface over which the integral is to be evaluated, unprimed variables are given in observation coordinates, primed variables are in source coordinates and the superscript 'h' denotes that the integral is hypersingular.

It should be noted that the integral in (1) is a dyadic with four terms. Each term is represented by I_{ij} and is given by,

$$I_{ij}^h(x_1,x_2) = \frac{\partial}{\partial x_i}\iint_{S'} \frac{\partial}{\partial x_j'} \frac{1}{\sqrt{(x_1-x_1')^2+(x_2-x_2')^2}} dx_1' dx_2' \qquad (2)$$

for $i, j = 1,2$

The first derivative in (2) is deliberately kept out of the integral sign and it can be taken under the integral sign only if the integral can be interpreted as a finite part integral. However the derivative under the integral sign can be converted to a derivative in observation coordinates by using $\partial/\partial x_i' = -\partial/\partial x_i$ and then can be taken outside the integral sign so that we can rewrite (2) as,

$$I_{ij}^h(x_1,x_2) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\iint_{S'} \frac{-1}{\sqrt{(x_1-x_1')^2+(x_2-x_2')^2}} dx_1' dx_2' \qquad (3)$$

The implication of (3) is that we can first evaluate the weakly singular integral and then apply numerical derivation to obtain the result for the hypersingular integral. Thus to evaluate the hypersingular integral, we start with numerical evaluation of the weakly singular integral,

$$I^w(x_1,x_2) = \iint_{S'} \frac{-1}{\sqrt{(x_1-x_1')^2+(x_2-x_2')^2}} dx_1' dx_2' \qquad (4)$$

Then we use second order central finite difference to obtain the elements of the dyadic given in (2). These elements are given as,

$$I_{11}^h(x_1,x_2) \cong \{I^w(x_1+\varepsilon,x_2) + I^w(x_1-\varepsilon,x_2) - 2I^w(x_1,x_2)\}/\varepsilon^2 \qquad (5a)$$

$$I_{22}^h(x_1,x_2) \cong \{I^w(x_1,x_2+\varepsilon) + I^w(x_1,x_2-\varepsilon) - 2I^w(x_1,x_2)\}/\varepsilon^2 \qquad (5b)$$

$$I_{21}^h(x_1,x_2) \equiv I_{12}^h(x_1,x_2) \cong \{I^w(x_1+\varepsilon,x_2+\varepsilon) + I^w(x_1-\varepsilon,x_2-\varepsilon) - I^w(x_1-\varepsilon,x_2+\varepsilon) - I^w(x_1+\varepsilon,x_2-\varepsilon)\}/4\varepsilon^2 \qquad (5c)$$

The error in evaluation of hypersingular singular integrals in (5) is due to three main sources. First the weakly singular integral in (4) is evaluated numerically and this introduces an error which depends on the type of the quadrature rules used as well as the number of quadrature nodes employed. Secondly, the centered difference formula for second order derivative is only second order accurate and introduces an error with order $O(\varepsilon^2)$. This is referred to truncation error, which is minimized by selecting ε arbitrarily small. A third source of error is rounding error which is also introduced by numerical derivation. The nominators in (5) can at most be evaluated to machine precision and the result is rounded at a specified digit. The error due to rounding is amplified by dividing the result to ε^2, since ε is selected as a small number to avoid truncation error.

In this study we selected the integration domain as a flat quadrilateral patch, which is a common meshing element in numerical simulations. The vertex points of the quarilateral are at wee (x_{1i}, x_{2i}) for i=1,2,4 and the observation point is at (x_{10}, x_{20}). This is plotted in Fig. 1. The surface is divided into four subtriangles sharing their common vertex at the singular point. It should be noted that the method is valid to any type of flat surface which can be represented as a collection of subtriangles.

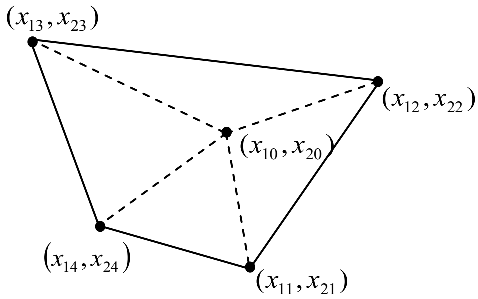

Fig. 1. The triangular patch over which the weakly singular integral is evaluated

In order to evaluate all the hypersingular integrals in (5) the weakly singular integral is evaluated at nine different points within the vicinity of the singular point. To evaluate the weakly singular integral we utilized the generalized cartesian product rule based on double exponential formula, which is shown to yield very accurate results for weakly singular integrals in [4]. The rule approximation of the quadrature rule is given by,

$$\int_{-1}^{1} f(u)du \cong h\sum_{k=-n}^{n} f(u_k)w_k \qquad (6)$$

where the product nh is kept constant, u_k are the nodes of the quadrature rule and w_k are the corresponding weights. The nodes are given by,

$$u_k = \tanh\left(\frac{\pi}{2}\sinh(hk)\right) \quad , k = [-n, n] \qquad (7)$$

and the weights w_k are given by,

$$w_k = \left[\frac{\pi}{2}\cosh(hk)\right] / \cosh^2\left(\frac{\pi}{2}\sinh(hk)\right) \quad , k = [-n, n] \qquad (8)$$

The advantage of using the quadrature rule based on the double exponential (DE) formula over other quadrature rules is that the rule places most of the nodes close to the endpoints and therefore it is effective for integrals having endpoint singularities.

3. Numerical Results

In order to show the validity of the formulas in (5), we evaluated the hypersingular integrals on a square patch with sidelength of 1 and whose center point is located at $(0.5,0.5)$. The observation point is selected at an arbitrary point within the triangle and is located at $(0.3,0.44)$. First, we evaluated the weakly singular integral in (4) by the quadrature rule employing double exponential formula and compared the result with analytical result obtained using Duffy transform [13]. In application of the rule we selected the product $nh = 3$, to avoid numerical underflow and overflow [4]. We have also used Gauss-Legendre quadrature rules to evaluate the same integral to allow comparison. The error of the quadrature rules, as a function of number of nodes, is plotted in Fig. 2.

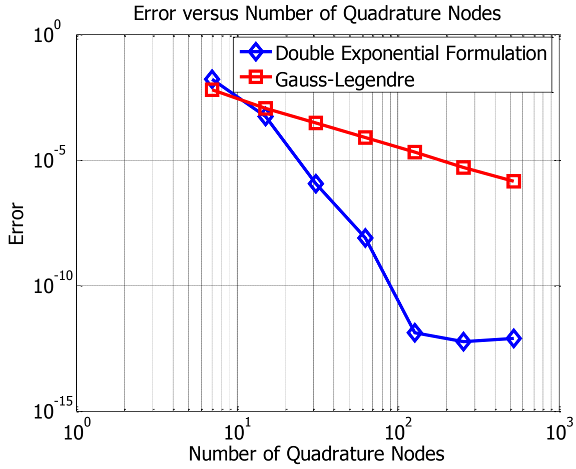

Fig. 2. Number of Nodes versus Error for quadrature rules, Double exponential formulation and Gauss-Legendre

It can be observed from Fig. 2 that quadrature rules employing double exponential formula give more accurate results for high number of quadrature nodes, when compared to Gauss-Legendre quadrature rules. Moreover the lower limit for DE formulation is around 10^{-12} which is obtained using about 127 quadrature nodes.

Next we evaluated the hypersingular integrals over the same patch using (5) and evaluated the error by comparing the numerical results with analytical results, which are obtained using [8]. We have presented the numerically and analytically obtained values in Table 1. Also we plotted the error as a function of ε in Fig. 3. Here we employed 127 point quadrature nodes based on DE formulation to evaluate weakly singular integrals.

Table 1. Results obtained for hypersingular integrals evaluated with 127 quadrature nodes with $\varepsilon = 10^{-4}$

	This Study (Numerical)	Reference [8] (Analytical)
$-I_{xx}$	7.345024958	7.345024270
$-I_{xy}$	0.2097641838	0.2097641264
$-I_{yy}$	5.442148228	5.442146051

From Fig. 3 it can be deduced that there is an optimum value for epsilon for which the best accuracy is obtained. Below this value rounding error dominates the total error and above this value truncation error is dominant.

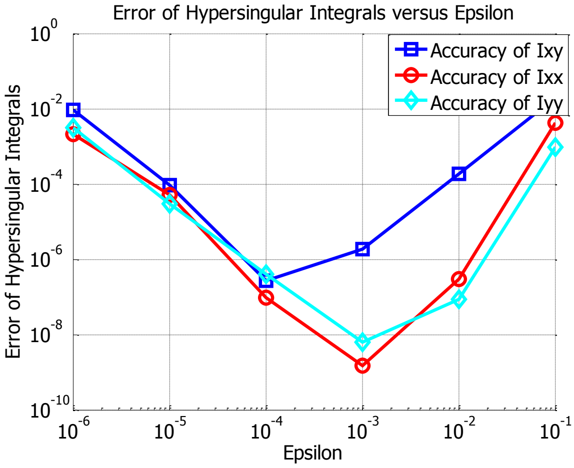

Fig. 3. The Accuracy of the Numerical Results of (5) as a function of epsilon

As the second numerical example we consider a TM^z scattering from a perfect electrically conducting (PEC) circular cylinder. The length of the cylinder is sufficiently long such that we can use Mie series solution as the analytical solution. The excitation is a plane wave with $E_0 = 120\pi \exp(jk_0 x)$ which is propagating in $-x$ direction. The radius 'a' of cylinder is selected such that we have $ka = 3$. The circumference of the cylinder is represented with 100 meshes and locally corrected Nyström method under one point quadrature rule is used to evaluate induced current on the cylinder. The current as a function of the angle around the cylinder is plotted in Fig. 4. It is observed from the figure that the numerical results are coherent with the analytical results obtained using the Mie series solution.

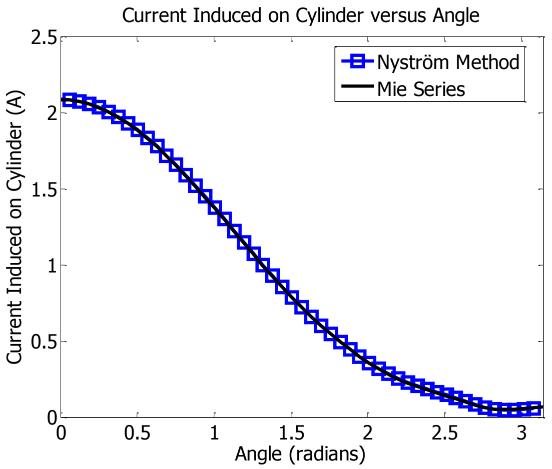

Fig. 4. Current induced on the cylinder for the TM^z problem

4. Conclusions

In this study we propose a new method for evaluation of hypersingular surface integrals. The method relies on the second order numerical derivative of the weakly singular surface integral. The weakly singular integral is evaluated using a quadrature rule based on the DE formula and numerical results have shown that this quadrature rule is far more accurate for weakly singular integral as compared to conventional quadrature rules such as Gauss-Legendre.

In order to evaluate the hypersingular integral second order derivative of the weakly singular integral is approximated by central difference equations. By numerical results it has been shown that both rounding error and truncation error degrade the accuracy for the results of hypersingular integrals.

Finally the locally corrected Nyström method is applied for the solution of a TM^z scattering problem from a PEC circular solution. In evaluating the self cell contribution, we have used the formulas in (5) and evaluated the current induced on the cylinder. Numerical results appear to be consistent with analytical results.

The new procedure is also applicable to curvilinear surface elements and we expect that this method will improve the error controlling capacity of Nyström method as well. The application of the method to non-planar surfaces is considered as a future study.

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