Obtaining the Time Response of Control Systems with Fractional Order PID from Frequency Responses

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Abstract

The paper deals with obtaining the time response of closed loop control system with fractional order PID controller using frequency response data. For this aim, a feedback control system with an integer order plant and a fractional order PID controller are studied. The real and imaginary parts of the closed loop transfer function are obtained which depend on the parameters K_p, K_i, K_d, λ and μ of fractional order PID controller and real and imaginary parts of the plant. Then the time domain responses of the closed loop control system with fractional order PID controller are plotted by using Inverse Fourier Transform Method (IFTM) or Fourier Series Method (FSM). The presented idea is supported by some numerical examples.

1. Introduction

Recently, there is a considerable interest in the fractional order system. Therefore, the number of studies on fractional order systems has been increasing. A system represented by differential equations where the orders of derivatives can take any real number, not necessarily integer number can be defined as a fractional order system. When the Laplace transform of such a system is taken, a fractional order transfer function (FOTF) is obtained [1-3]. It includes fractional order Laplace complex variable s such as s^{α}, $\alpha \in R$. Due to the lack of analytical solutions of such systems, computing the time response of a FOTF is a difficult problem. This problem has been usually overcome by integer approximation methods such as Continued Fractional Expansion (CFE) method, Oustaloup's method, Carlson's method, Matsuda's method, Chareff's method, least square methods [4-10] and numerical approximation methods such as Grünwald-Letnikov approximation [2] are used for step and impulse response computations. Additionally, it is possible to come across some methods such as Mittag-Leffler function and Gamma function [10, 11].

However, in a recent paper by authors, the computation problem of inverse Laplace transform of fractional order systems was solved by using two methods [12, 13]. One is based on Fourier series of a square wave and therefore it is called Fourier Series Method (FSM) and the other is based on Inverse Fourier Transform Method (IFTM). Because of using exact frequency response data such as gain and phase data in FSM and IFTM methods, the results obtained from these methods are exact. It is well known that the frequency response of a FOTF can be computed exactly by using $(j\omega)^{\alpha} = \omega^{\alpha}[\cos(\alpha\pi/2) + j\sin(\alpha\pi/2)]$.

In this paper, we benefited from these methods for time response computation of control systems with a fractional order PID controller. A closed loop control system with a plant and controller can be in the form of four different situations. Respectively, both plant and controller are integer order, both plant and controller are fractional order, controller is fractional order and plant is integer order, controller is integer order and plant is fractional order [6]. Generally, a model of any real system can be obtained approximately as integer order. However a fractional order control system designed better instead of a classical integer order controller can be more advantages. Therefore, here, we studied third situation where the controller is a fractional order PID controller such as $C(s) = K_p + K_i / s^{\lambda} + K_d s^{\mu}$ and plant is an integer order transfer function. Thus, the papers given in [12, 13] are used to find the unit step and unit impulse responses of the closed loop system with an integer order plant and a fractional order PID controller. Time response equations which depend on the controller parameters are derived using real and imaginary part of the transfer function of the plant. Since the frequency response of a FOTF can be obtained exactly, the time responses obtained from presented methods are accurate results.

The paper is organized as follows: In Section 2, a brief review of fractional order system is given and a closed loop control system with fractional order PID controller is introduced. The exact methods based on FSM and IFTM for computation of step and impulse responses of closed loop control systems with fractional order PID controller are introduced in Section 3. Examples are given to show the importance of the method presented in Section 4. Concluding remarks are given in Section 5.

2. Control Systems with Fractional Order PID Controller

The study of non integer order derivatives and integrals is known as fractional calculus. Many mathematicians like Liouville and Riemann dealt with the field of fractional calculus and presented many publications. Many different definitions can be found such as Grünwald-Letnikov, Riemann-Lioville and Caputo for fractional order operators [10]. In recent years, it is possible to come across a lot of studies related to fractional order system such as stability analysis, controller design, frequency response and time response analysis of control systems [1-3]. These studies showed that fractional order PI and PID controllers instead of classical controllers can give better results [6, 11, 14]. A closed loop control system with a fractional order PID controller is shown in Fig. 1.

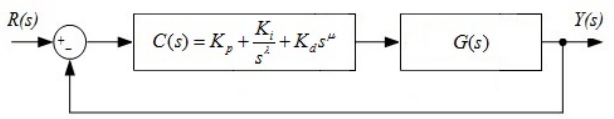

Fig. 1. A closed loop control system with a fractional order PID controller

In Fig.1, the fractional order PID controller is in the form of

$$C(s) = K_p + \frac{K_i}{s^\lambda} + K_d s^\mu \tag{1}$$

and the plant model which is an integer order transfer function is shown in Eq. (2).

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \tag{2}$$

Using Eqs. (1) - (2), the closed loop transfer function of the system can be written as

$$P(s) = \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\left(K_p + \frac{K_i}{s^\lambda} + K_d s^\mu\right) G(s)}{1 + \left(K_p + \frac{K_i}{s^\lambda} + K_d s^\mu\right) G(s)} \tag{3}$$

Replacing s by $j\omega$ and using Eq. (4) and Eq. (5) in Eq. (3), one obtains Eq. (6)

$$(j\omega)^\lambda = \omega^\lambda [\cos(\lambda\pi / 2) + j \sin(\lambda\pi / 2)] \tag{4}$$

$$(j\omega)^\mu = \omega^\mu [\cos(\mu\pi / 2) + j \sin(\mu\pi / 2)] \tag{5}$$

$$P(j\omega) = \frac{\left(K_p + \frac{K_i}{(j\omega)^\lambda} + K_d (j\omega)^\mu\right) G(j\omega)}{1 + \left(K_p + \frac{K_i}{(j\omega)^\lambda} + K_d (j\omega)^\mu\right) G(j\omega)} \tag{6}$$

Eq. (6) is also shown as Eq. (7) which depends on $M(\omega)$, $N(\omega)$, $Q(\omega)$ and $Z(\omega)$ functions.

$$P(j\omega) = \frac{M(\omega) + jN(\omega)}{Q(\omega) + Z(\omega)} \tag{7}$$

$$G(j\omega) = \mathrm{Re}[G(j\omega)] + j\,\mathrm{Im}[G(j\omega)] \tag{8}$$

$M(\omega)$, $N(\omega)$, $Q(\omega)$ and $Z(\omega)$ functions are obtained depending on the parameters of $C(s)$ and $G(s)$ as below

$$\begin{aligned} M(\omega) &= \mathrm{Re}[G(j\omega)][K_p + K_i \cos(\lambda\tfrac{\pi}{2}) / \omega^\lambda + K_d \omega^\mu \cos(\mu\tfrac{\pi}{2})] \\ &+ \mathrm{Im}[G(j\omega)][K_i \sin(\lambda\tfrac{\pi}{2}) / \omega^\lambda - K_d \omega^\mu \sin(\mu\tfrac{\pi}{2})] \end{aligned} \tag{9}$$

$$\begin{aligned} N(\omega) &= \mathrm{Im}[G(j\omega)][K_p + K_i \cos(\lambda\tfrac{\pi}{2}) / \omega^\lambda + K_d \omega^\mu \cos(\mu\tfrac{\pi}{2})] \\ &+ \mathrm{Re}[G(j\omega)][-K_i \sin(\lambda\tfrac{\pi}{2}) / \omega^\lambda + K_d \omega^\mu \sin(\mu\tfrac{\pi}{2})] \end{aligned} \tag{10}$$

$$\begin{aligned} Q(\omega) &= 1 + \mathrm{Re}[G(j\omega)][K_p + K_i \cos(\lambda\tfrac{\pi}{2}) / \omega^\lambda + K_d \omega^\mu \cos(\mu\tfrac{\pi}{2})] \\ &+ \mathrm{Im}[G(j\omega)][K_i \sin(\lambda\tfrac{\pi}{2}) / \omega^\lambda - K_d \omega^\mu \sin(\mu\tfrac{\pi}{2})] \end{aligned} \tag{11}$$

$$\begin{aligned} Z(\omega) &= \mathrm{Im}[G(j\omega)][K_p + K_i \cos(\lambda\tfrac{\pi}{2}) / \omega^\lambda + K_d \omega^\mu \cos(\mu\tfrac{\pi}{2})] \\ &+ \mathrm{Re}[G(j\omega)][-K_i \sin(\lambda\tfrac{\pi}{2}) / \omega^\lambda + K_d \omega^\mu \sin(\mu\tfrac{\pi}{2})] \end{aligned} \tag{12}$$

$$P(j\omega) = \mathrm{Re}[P(j\omega)] + j\,\mathrm{Im}[P(j\omega)] \tag{13}$$

Now, one can compute $\mathrm{Re}[P(j\omega)]$ and $\mathrm{Im}[P(j\omega)]$ as follows

$$\mathrm{Re}[P(j\omega)] = \frac{[M(\omega)Q(\omega) + N(\omega)Z(\omega)]}{[Q(\omega)^2 + Z(\omega)^2]} \tag{14}$$

$$\mathrm{Im}[P(j\omega)] = \frac{[N(\omega)Q(\omega) - M(\omega)Z(\omega)]}{[Q(\omega)^2 + Z(\omega)^2]} \tag{15}$$

3. Time Response of Fractional Order Control Systems

In this section, Fourier Series Method (FSM) and Inverse Fourier Transform Method (IFTM) previously given in [12] are used for time domain computation of the closed loop control system shown in Fig. 1.

3.1. Fourier Series Method (FSM)

The Fourier series for the square wave of -1 to 1 with frequency $\omega_s = 2\pi / T$ can be written as

$$r(t) = \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \frac{1}{k} \sin(k\omega_s t) \tag{16}$$

where T is the period of the square wave. If $r(t)$ is the input of the control system given in Fig. 1 means it passes through the transfer function $P(s)$ then the output, which is the unit step response if T is sufficiently large, can be written as

$$y_s(t) = \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \left(\begin{array}{l} \frac{1}{k} \mathrm{Re}\left[P(jk\omega_s)\right] \sin(k\omega_s t) \\ + \frac{1}{k} \mathrm{Im}[P(jk\omega_s)] \cos(k\omega_s t) \end{array} \right) \tag{17}$$

As $T \to \infty$ and $\omega_s \to 0$ the numerator of the imaginary part of $P(jk\omega_s)$ is multiplied by ω_s so that $\lim_{\omega_s \to 0} \mathrm{Im}\, P(jk\omega_s) = 0$ and Eq. (17) becomes

$$y_s(t) \cong \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \frac{1}{k} \mathrm{Re}\left[P(jk\omega_s)\right] \sin(k\omega_s t) \quad (18)$$

Using Eq. (14) , this equation can be written as

$$y_s(t) \cong \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \frac{1}{k} \frac{[M(k\omega_s)Q(k\omega_s) + N(k\omega_s)Z(k\omega_s)]}{[Q(k\omega_s)^2 + Z(k\omega_s)^2]} \sin(k\omega_s t) \quad (19)$$

which is the unit step response of $P(s)$. Similarly, the impulse response, which is the derivative of the step response, is given by

$$\begin{aligned} y_i(t) &= \frac{dy_s(t)}{dt} = \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \begin{pmatrix} \omega_s \mathrm{Re}\left[P(jk\omega_s)\right] \cos(k\omega_s t) \\ -\omega_s \mathrm{Im}[P(jk\omega_s)] \sin(k\omega_s t) \end{pmatrix} \\ &\cong \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \omega_s \mathrm{Re}\left[P(jk\omega_s)\right] \cos(k\omega_s t) \end{aligned} \quad (20)$$

Similarly, using Eq. (14), Eq. (20) can be written as

$$y_i(t) \cong \frac{4}{\pi} \sum_{k=1(2)}^{\infty} \omega_s \frac{[M(k\omega_s)Q(k\omega_s) + N(k\omega_s)Z(k\omega_s)]}{[Q(k\omega_s)^2 + Z(k\omega_s)^2]} \cos(k\omega_s t) \quad (21)$$

Thus, the step and impulse responses of $P(s)$ can be obtained respectively using Eqs. (19) and (21).

3.2. Inverse Fourier Transform Method (IFTM)

The impulse response, $g(t)$, corresponding to the transfer function $G(s)$ is given by $g(t) = L^{-1}(G(s))$ where L^{-1} denotes the inverse Laplace transform. Assuming the impulse response is that of a stable system so that $\lim_{t\to\infty} g(t) = 0$ then the Fourier transform can be evaluated.

It has been shown in [12] that for a transfer function $G(s)$ the unit impulse response, $g(t)$, can be evaluated from

$$g(t) = \frac{2}{\pi} \int_0^{\infty} \mathrm{Re}[G(j\omega)] \cos(\omega t) d\omega \quad (22)$$

or

$$g(t) = -\frac{2}{\pi} \int_0^{\infty} \mathrm{Im}[G(j\omega)] \sin(\omega t) d\omega \quad (23)$$

Now, unit impulse response for the closed loop fractional order control system given in Fig. 1 can be computed from the closed loop transfer function, $P(s)$, of the system. For this case, Eqs. (22) and (23) can be written as

$$p(t) = \frac{2}{\pi} \int_0^{\infty} \mathrm{Re}[P(j\omega)] \cos(\omega t) d\omega \quad (24)$$

or

$$p(t) = -\frac{2}{\pi} \int_0^{\infty} \mathrm{Im}[P(j\omega)] \sin(\omega t) d\omega \quad (25)$$

Substituting, Eqs. (14) and (15) into Eqs. (24) and (25) respectively, one can obtain the following equations.

$$p(t) = \frac{2}{\pi} \int_0^{\infty} \frac{[M(\omega)Q(\omega) + N(\omega)Z(\omega)]}{[Q(\omega)^2 + Z(\omega)^2]} \cos \omega t d\omega \quad (26)$$

$$p(t) = -\frac{2}{\pi} \int_0^{\infty} \frac{[N(\omega)Q(\omega) - M(\omega)Z(\omega)]}{[Q(\omega)^2 + Z(\omega)^2]} \sin \omega t d\omega \quad (27)$$

Thus, $p(t)$ can be computed by numerical integration using Eq. (26) or Eq. (27). Matlab has several functions available for integration and **trapz** has been used in this paper for evaluating the integration.

4. Examples

4.1. Example 1

The aim of this example is to prove the validity of presented method by considering the control system of Fig. 1 with integer order transfer functions such as

$$C(s) = 18 + \frac{13}{s} + 6s \text{ and } G(s) = \frac{1}{s(s+1)(s+5)} \quad (28)$$

The step responses of the system obtained by Matlab step function and the FSM program are shown in Fig. 2 where it can be seen that both results are same. And also, it can be seen in Fig. 3 that the maximum error between two plots was computed as $3.423x10^{-5}$. Similarly, the impulse responses of the closed loop system using Matlab impulse function and FSM are plotted in Fig. 4 where it was estimated that the maximum error between two plots is $1.027x10^{-4}$. These errors occur at initial time and then the error between two plots becomes zero as seen in Fig. 3 and Fig. 5. Also, the initial time errors ($3.423x10^{-5}$ and $1.027x10^{-4}$) are very small. Therefore, the presented method gives exact step and impulse responses of the closed loop system since the approach is based on using the frequency response data of the system which can be computed exactly.

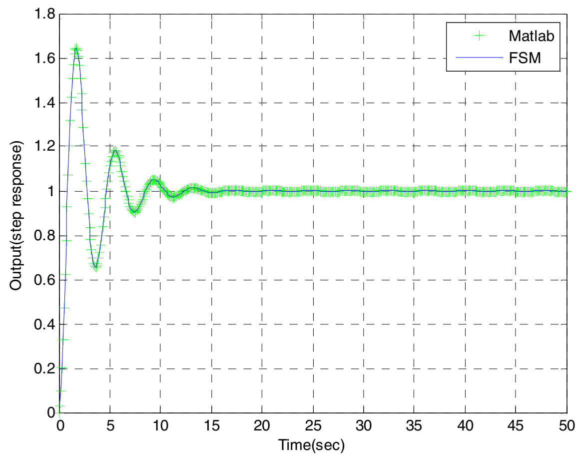

Fig. 2. Step responses

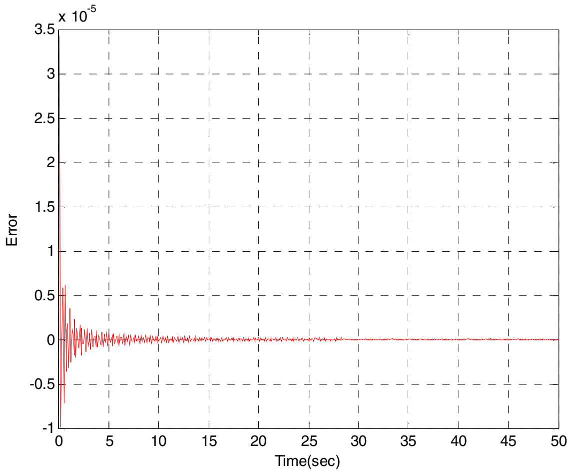

Fig. 3. Error between Matlab and FSM for step responses

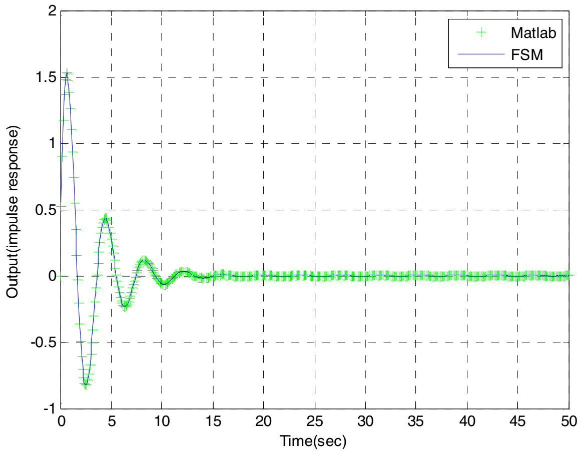

Fig. 4. Impulse responses

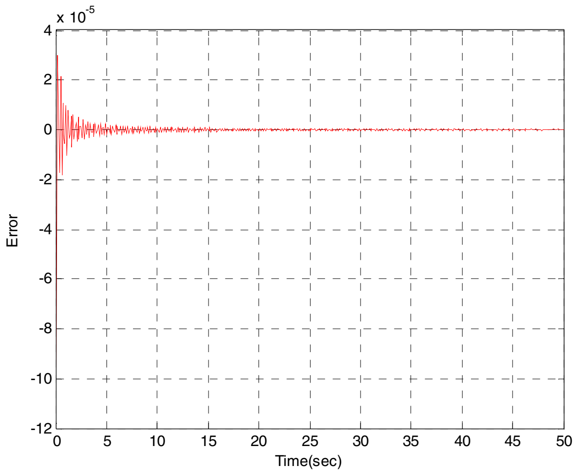

Fig. 5. Error between Matlab and FSM for impulse responses

4.2. Example 2

Consider Fig. 1 with fractional order PID controller and integer order plant transfer function below.

$$C(s) = 18 + \frac{13}{s^{0.8}} + 6s^{1.4} \text{ and } G(s) = \frac{1}{s(s+1)(s+5)} \tag{29}$$

The open loop transfer function is

$$L(s) = C(s)G(s) = \frac{6s^{2.2} + 18s^{0.8} + 13}{s^{3.8} + 6s^{2.8} + 5s^{1.8}} \tag{30}$$

and the closed loop transfer function is

$$\begin{aligned} P(s) &= \frac{L(s)}{1+L(s)} = \frac{6s^{2.2} + 18s^{0.8} + 13}{s^{3.8} + 6s^{2.8} + 6s^{2.2} + 5s^{1.8} + 18s^{0.8} + 13} \\ &= \frac{6s^2(s^{0.2}) + 18s^{0.8} + 13}{s^3(s^{0.8}) + 6s^2(s^{0.8}) + 6s^2(s^{0.2}) + 5s(s^{0.8}) + 18s^{0.8} + 13} \end{aligned} \tag{31}$$

Here, comparison results with Oustaloup method and Grünwald-Letnikov (GL) are given. Oustaloup's fifth order integer approximations for $s^{0.2}$ and $s^{0.8}$ are

$$s^{0.2} \cong \frac{2.512s^5 + 98.83s^4 + 531.7s^3 + 442.3s^2 + 56.87s + 1}{s^5 + 56.87s^4 + 442.3s^3 + 531.7s^2 + 98.83s + 2.512} \tag{32}$$

$$s^{0.8} \cong \frac{39.81s^5 + 901.4s^4 + 2790s^3 + 1336s^2 + 98.83s + 1}{s^5 + 98.83s^4 + 1336s^3 + 2790s^2 + 901.4s + 39.81} \tag{33}$$

Using these approximations in Eq. (31), one can obtain 13th order closed loop transfer function.

The exact step response of the system obtained from FSM, the step response of Oustaloup 5th order and step response of Grunwald-Letnikov (GL) method are shown in Fig 6. It can be seen in Fig. 7 that the difference between Oustaloup 5th order and FSM is very small. However, Grunwald-Letnikov error is a little big than Oustaloup and FSM. We used the GL program [2] for Grunwald-Letnikov step response computation.

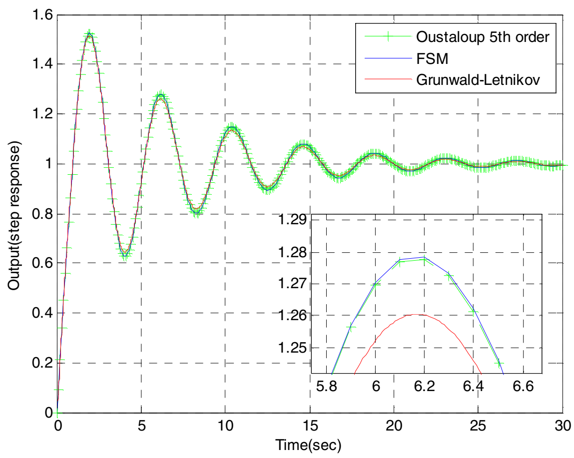

Fig. 6. Step responses for fractional order PID controller

The step response of $P(s)$ using FSM and the impulse responses of $\frac{1}{s}P(s)$ which is the step response of $P(s)$ using IFTM are plotted in Fig. 8. From Fig. 8, one can see that the two plots fit exactly.

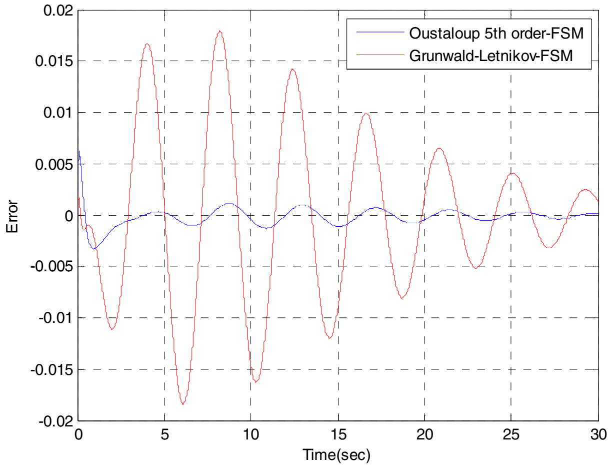

Fig. 7. Error between Oustaloup 5[th] order and Grunwald-Letnikov according to FSM for step responses

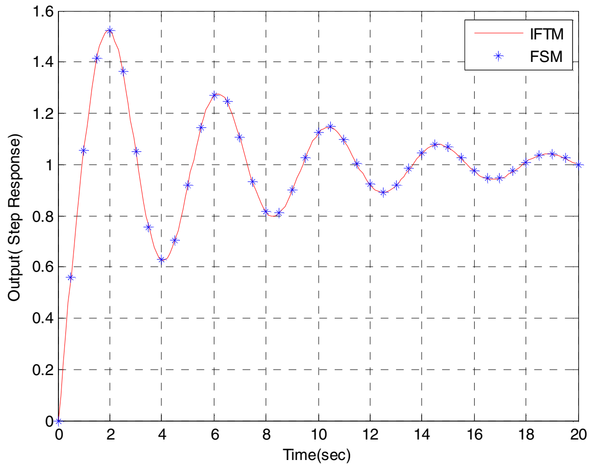

Fig. 8. Step responses obtained from FSM and IFTM

5. Conclusions

Two exact methods have been presented for the time response computation of the closed loop control systems with fractional order PID controller. The methods are based on using the frequency response data of the closed loop fractional order control system. It has been shown that the unit step and unit impulse responses of a feedback control system including a fractional order PID controller can be computed exactly using Fourier series of a square wave and inverse Fourier transform of frequency response information namely gain and phase data. Time response equations which are the function of controller parameters have been obtained. Given examples clearly show that the presented methods provide very useful results in the field of fractional order control systems. Especially the results will be very attractive for the design of fractional order control systems.

6. Acknowledgement

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