Fractional order PID controller design based on Laguerre orthogonal functions

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Abstract

This paper proposes a novel fractional order PID controller for fractional order systems based on Laguerre orthonormal functions. The transfer functions of the fractional order plant, the desired loop gain and the fractional order PID controller are expanded in terms of their Laguerre basis functions. Matching the first three coefficients of the Laguerre series of the loop gain with the desired one yields the fractional order PID controller parameters. The pole of the fractional order Laguerre basis function is adjusted to minimize an integral square error performance index subject to control signal constraint. The numerical example is presented to show the effectiveness of this Laguerre based fractional order PID controller, as well.

1. Introduction

Fractional calculus concerns utilizing non-integer derivatives and integrals instead of the corresponding ordinary ones to increase the design flexibility and modelling precision [1]. The Fractional Order PID (FOPID) controllers with fractional order derivative and integral terms have been employed to control fractional order systems. A lot of approaches have been proposed to design FOPID controllers for fractional order systems in the literature. Internal model based FOPID controllers have been considered in this regard [2]. Designing fractional order PD (FOPD) controllers robust to gain variations has been considered, too [3]. Designing these controllers to minimize a performance index based on optimization methods is another approach [4]. Root locus method has been employed to design FOPID for minimum-phase fractional order systems [5].

One of the analytical approaches proposed for designing PID controllers is the moment matching method. In this method, the PID controller parameters could be obtained by matching the first three moments of the closed loop transfer function with the desired one. In this approach, the closed loop transfer function is expanded in terms of some orthogonal functions. For example, the MacLaurin expansion has been employed to find the PID controller parameters through a moment matching approach [6]. A PD controller for the first-order plus dead time plants is designed based on the Taylor series approximation, too [7]. Laguerre orthogonal functions have been utilized to design PID controllers for some special case of plants [8]. For FOPID controllers, a moment matching method has been proposed in the literature [9]. In the proposed approach, the first three moments of the desired closed loop transfer function obtained from a characteristic ratios assignment approach are matched with the corresponding ones in the closed loop transfer function. The proposed method could be employed to design FOPID for commensurate order fractional systems. Block pulse, Walsh and Haar Wavelet as piecewise orthogonal functions have been employed to design FOPID for integer and fractional order systems, too [10].

Fractional order Laguerre orthogonal functions have been constructed as a generalization of ordinary Laguerre functions [11]. The mentioned fractional order Laguerre functions have been employed to approximate fractional order systems [12].

In the current paper, the fractional order Laguerre expansion is utilized to synthesis FOPID and Fractional order PI (FOPI) controllers for fractional order systems. First, a commensurate order fractional system is interpreted in terms of its fractional order Laguerre series. The Laguerre series coefficients are obtained by the inner product of the plant transfer function with the fractional order Laguerre basis functions. This idea is employed to calculate the Laguerre series coefficients of the loop gain (the product of the FOPID controller and the plant). Matching the first three Laguerre series coefficients of the loop gain with the desired one gives the FOPID or FOPI controller parameters. The optimum location of the fractional order Laguerre basis function pole is determined so that the best fitting to the desired loop gain is achieved. The performance of this Laguerre based FOPID or FOPI controller is investigated through numerical simulations.

The remainder of this paper is organized as follows. A brief review on fractional calculus is given in Section 2. Section 3 describes the construction of the fractional order Laguerre series basis functions. The proposed FOPID and FOPI controller are presented in Section 4. The performance of the proposed FOPID and FOPI controllers is demonstrated by numerical simulation given in Section 5. Finally, Section 6 concludes the paper.

2. Fractional calculus definitions

There are a lot of definitions for the fractional order derivative in the literature [1]. Due to its computation advantages, the Grunwald-Letnikov definition is utilized in this paper. According to this definition, the fractional order derivative of a function $f(t)$ ($D^{\rho} f(t)$) is defined as [1]

$$D^{\rho} f(t) = \lim_{h \to 0} h^{-\rho} \sum_{j=0}^{[\frac{t}{h}]} (-1)^j \binom{\rho}{j} f(t - jh), \quad (n-1 < \rho \le n) \quad (1)$$

where ρ is the fractional order. If $\rho = 1$, then the ordinary definition of derivative is obtained. The Fractional Order Transfer Function (FOTF) toolbox proposed for numerical simulation of fractional order systems is based on this definition [13]. A Linear Time Invariant (LTI) fractional order system with input $u(t)$ and output $y(t)$ could be described with the following differential equation

$$a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + ... + a_0 D^{\alpha_0} y(t) = \\ b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + ... + b_0 D^{\beta_0} u(t) \quad (2)$$

where $\alpha_i (i = 0,...,n)$ and $\beta_j (j = 0,...,m)$ are the fractional orders and $a_k (k = 0,...,n)$ and $b_k (k = 0,...,m)$ are arbitrary constant numbers. If $\alpha_i = iv, i = 1,...,n$ and $\beta_j = jv, j = 1,...,m$ are considered, then the fractional order system (2) is called commensurate and v is called the commensurate order. Taking Laplace transform from both sides of (2) gives the transfer function of a commensurate order fractional system as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{mv} + b_{m-1} s^{(m-1)v} + ... + b_0}{a_n s^{nv} + a_{n-1} s^{(n-1)v} + ... + a_0}. \quad (3)$$

The controllers containing fractional order operators in their structure are called fractional order controllers. For example, the transfer function of a fractional order PID controller could be written as

$$C(s) = k_c (1 + \frac{1}{T_i s^{\beta}} + T_d s^{\mu}) \quad (4)$$

where k_c , T_i and T_d are the proportional gain, integrator and derivative coefficients, respectively. While β and μ are two arbitrary real numbers belonging to $(0,2)$. If $\beta = \mu = 1$, then the ordinary PID controller is obtained.

3. Fractional order Laguerre orthogonal functions

In this section, fractional order Laguerre basis functions are introduced. To begin, some necessary preliminaries should be defined. In [14], it is shown that the fractional order transfer function (3) is stable if the following conditions are satisfied

$$0 < v < 2, \quad |\arg(z)| < \frac{v\pi}{2}, \quad a_n z^n + a_{n-1} z^{n-1} + ... + a_0 = 0 \,. \quad (5)$$

Moreover, the transfer function (3) belongs to $H_2(C^+)$ if the following inequality is fulfilled [11]

$$(n - m)v > 0.5 \,. \quad (6)$$

To construct the fractional order Laguerre basis functions, the following generating functions are defined

$$F_n(s) = \frac{1}{(s^v + \lambda)^n} \,. \quad (7)$$

According to (6), the generating functions in (7) belong to $H_2(C^+)$, if the following inequality holds

$$n \ge n_0, \quad n_0 = \left[\frac{1}{2v}\right] + 1 \,. \quad (8)$$

According to (8), the transfer function (7) belongs to $H_2(C^+)$ for all $n \ge 1$ ($n_0 = 1$) if $v \in (0.5, 2)$. Thus, in the remainder of the paper, the transfer function of the plant is considered a commensurate order fractional system as (3) with commensurate order $v \in (0.5, 2)$. Unfortunately, the generating functions (7) aren't orthogonal and couldn't be utilized directly as Laguerre basis functions [11]. Therefore, these functions are employed to generate fractional order Laguerre basis functions according to a Gram-Schmidt orthogonalization procedure [11].

Gram-Schmidt orthogonalization procedure: consider arbitrary generating functions $F_i(s) \in H_2(C^+)$ $, i = 1, \ldots, N$. Now, the functions $\varphi_i(s) \in H_2(C^+)$ $, i = 1, \ldots, N$ obtained from the following relations are orthonormal

$$\varphi_1(s) = \frac{F_1(s)}{\|F_1(s)\|}, \\ \varphi_i(s) = \frac{F_i(s) - \sum_{j=1}^{i-1} < F_i(s), \varphi_j(s) > \varphi_j(s)}{\left\| F_i(s) - \sum_{j=1}^{i-1} < F_i(s), \varphi_j(s) > \varphi_j(s) \right\|}, \quad i = 1, \ldots, N \quad (9)$$

where $< p, q >$ denotes the inner product of functions p and q defined as follows

$$< p, q > = \frac{1}{2\pi} \int_{-\infty}^{+\infty} p(j\omega) \overline{q(j\omega)} d\omega \,. \quad (10)$$

Moreover, the norm of function p is defined as

$$\|p\| = \sqrt{< p, p >} \,. \quad (11)$$

This yields the orthonormal fractional order Laguerre basis functions $\varphi_i(s) \in H_2(C^+)$ $, i = 1, \ldots, k$. To construct fractional order Laguerre basis functions, inner product of a fractional order plant with the generating functions (7) should be calculated. The transfer function of any commensurate order plant with real poles could be described with a partial fractions form in terms of pseud first order terms (7). Thus, it is enough to compute the inner product of two generating functions like (7)

$$\theta(v, h, n, \lambda, \delta) = < \frac{1}{(s^v + \lambda)^h}, \frac{1}{(s^v + \delta)^n} > = \\ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{((j\omega)^v + \lambda)^h (\overline{j^v} \omega^v + \sigma)^m} \,. \quad (12)$$

The relation (12) could be rewritten as

$$\theta(v, h, n, \lambda, \delta) = \frac{1}{2\pi} (\int_0^{+\infty} \frac{d\omega}{((j\omega)^v + \lambda)^h (\overline{j^v} \omega^v + \sigma)^m} + \\ \int_0^{+\infty} \frac{d\omega}{((-j\omega)^v + \lambda)^h (\overline{(-j)^v} \omega^v + \sigma)^m}) \,. \quad (13)$$

By change of variable $x = \omega^v$ equation (13) is simplified as

$$\theta(v,h,n,\lambda,\delta)=\frac{1}{2\pi v\lambda^h\delta^m}[I(\lambda^{-1}e^{j\frac{\pi}{2}v},\delta^{-1}e^{-j\frac{\pi}{2}v},\frac{1}{v},h,m) +I(\lambda^{-1}e^{-j\frac{\pi}{2}v},\delta^{-1}e^{j\frac{\pi}{2}v},\frac{1}{v},h,m)] \quad (14)$$

where

$$I(\zeta,\gamma,\frac{1}{v},h,m)=\int_0^\infty \frac{x^{1/v-1}dx}{(\zeta x+1)^h(\gamma x+1)^m}. \quad (15)$$

The details of computing integral (15) is illustrated in [3.194, 4 p.285] of [15].
Now, a commensurate order strictly proper system as (3) could be expanded as the following fractional order Laguerre series

$$G(s)=\sum_{i=1}^{\infty} g_i\varphi_i(s) \quad (16)$$

where $\varphi_i(s)$ are the Laguerre basis functions constructed from a Gram-Schmidt procedure. The Laguerre basis functions could be parameterized as follows

$$\varphi_i(s)=\sum_{j=1}^{i}\frac{s_{ij}}{(s^v+\lambda)^j} \quad (17)$$

where $s_{ij},\ j=1,\ldots,i,\ i=1,2,3,\ldots$ are constant parameters obtained from the Gram-Schmidt procedure. Moreover, the Laguerre series coefficients g_i are calculated as

$$g_i=<G(s),\varphi_i(s)>. \quad (18)$$

Example 1: Consider the following fractional order system

$$G(s)=\frac{2}{s^{0.7}+1}. \quad (19)$$

Considering the Laguerre pole $\lambda=2.5$, gives the unknown parameters of Laguerre basis functions in (17) as

$$\begin{aligned}&s_{11}=1.5037, s_{21}=-1.2422, s_{22}=10.8697,\\ &s_{31}=-1.159, s_{32}=22.6265, s_{33}=-71.9935\end{aligned}. \quad (20)$$

The first three Laguerre series coefficients of (19) are

$$g_1=0.8227,\quad g_2=0.2492,\quad g_3=0.0826. \quad (21)$$

To design Laguerre based FOPI and FOPID controllers, the product of each pairs of two Laguerre basis functions should be computed. By some manipulations, we have

$$\varphi_i(s)\varphi_j(s)=\sum_{k=1}^{i+j}a_{ijk}\varphi_k(s). \quad (22)$$

where the coefficients a_{ijk} could be obtained in terms of s_{ij} . for example, we have

$$a_{112}=\frac{s_{11}^2}{s_{22}},\quad a_{111}=-\frac{s_{11}s_{21}}{s_{22}}. \quad (23)$$

Other coefficients in (22) could be calculated similarly.

4. Laguerre based design of FOPI and FOPID controllers

The plant is considered as a stable commensurate order fractional system with the Laguerre expansion (16). The controller $C(s)$ is designed such that the loop gain could approximate a desired loop gain $L(s)$. Or:

$$L(s)=G(s)C(s). \quad (24)$$

In a unit negative feedback control structure, the following desired loop gain is considered

$$L(s)=\frac{\omega_n^2}{s^v(s^v+2\eta\omega_n)}. \quad (25)$$

where η and ω_n are the damping ratio and the natural frequency, respectively. The desired loop gain isn't a stable function. Therefore, it should be rewritten as follows

$$L(s)=\frac{\dfrac{\omega_n^2}{(s^v+2\eta\omega_n)(s^v+\lambda)^2}}{\dfrac{s^v}{(s^v+\lambda)^2}}=\frac{\sum_{i=1}^{\infty}l_i\varphi_i(s)}{\sum_{i=1}^{\infty}l'_i\varphi_i(s)} \quad (26)$$

where the first three Laguerre series coefficients in (26) are calculated by partial fraction expansion as

$$\begin{aligned}l_i=&<\frac{\omega_n^2}{(s^v+2\eta\omega_n)(\lambda-2\eta\omega_n)^2},\varphi_i(s)>+\\ &<\frac{-\omega_n^2}{(s^v+\lambda)(\lambda-2\eta\omega_n)^2},\varphi_i(s)>+\\ &<\frac{\omega_n^2}{(s^v+\lambda)^2(2\eta\omega_n-\lambda)^2},\varphi_i(s)>,\quad i=1,2,3\end{aligned}. \quad (27)$$

$$l'_1=\frac{s_{22}+\lambda s_{21}}{s_{11}s_{22}},\ l'_2=\frac{-\lambda}{s_{22}},\ l'_3=0. \quad (28)$$

The design procedure is illustrated in the following subsections

4.1. Laguerre based FOPI design

Consider the following FOPI controller

$$C(s)=k_c(1+\frac{1}{T_is^v}). \quad (29)$$

The controller could be rewritten in the following Laguerre series form

$$C(s)=\frac{\frac{T_i s^v+1}{T_i(s^v+\lambda)^2}}{\frac{1}{k_c}\frac{s^v}{(s^v+\lambda)^2}}=\frac{\sum_{i=1}^{\infty}c_i\varphi_i(s)}{\sum_{i=1}^{\infty}c_i'\varphi_i(s)}. \quad (30)$$

where

$$c_1=\frac{k_c T_i s_{22}-k_c s_{21}(-\lambda T_i+1)}{T_i s_{11}s_{22}},\quad c_2=\frac{k_c(1-\lambda T_i)}{T_i s_{22}},\quad c_1'=\frac{s_{22}+\lambda s_{21}}{s_{11}s_{22}},\quad c_2'=\frac{-\lambda}{s_{22}}. \quad (31)$$

Substituting relations (16), (26) and (30) in (24) yields

$$\sum_{i=1}^{\infty}g_i\varphi_i(s)\sum_{i=1}^{\infty}c_i\varphi_i(s)\sum_{i=1}^{\infty}l_i'\varphi_i(s^\gamma)=\sum_{i=1}^{\infty}l_i\varphi_i(s)\sum_{i=1}^{\infty}c_i'\varphi_i(s). \quad (32)$$

Considering the product property of fractional order Laguerre basis functions in (22) and matching the first two coefficients in both sides in series (32) gives the following FOPI coefficients

$$k_c=s_{11}c_1+s_{21}c_2,\quad T_i=\frac{s_{11}c_1+s_{21}c_2}{s_{22}c_2+\lambda(s_{11}c_1+s_{21}c_2)} \quad (33)$$

where

$$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}X_1 & X_2\\Y_1 & Y_2\end{bmatrix}^{-1}\begin{bmatrix}q_1\\q_2\end{bmatrix}. \quad (34)$$

$$X_i=\sum_{j=1}^{2}a_{ij1}u_j,\quad Y_i=\sum_{j=1}^{2}a_{ij2}u_j,\quad i=1,2. \quad (35)$$

$$u_i=\sum_{k=1}^{2}\sum_{j=1}^{2}g_j l_k' a_{jki},\quad q_i=\sum_{k=1}^{2}\sum_{j=1}^{2}l_j c_k' a_{jki},\quad i=1,2. \quad (36)$$

4.2. Laguerre based FOPID design

The FOPID controller with the following transfer function is considered

$$C(s)=k_c\left(1+\frac{1}{T_i s^v}+T_d s^v\right). \quad (37)$$

Transfer function (37) is rewritten as

$$C(s)=\frac{\frac{k_c(T_i T_d s^{2v}+T_i s^v+1)}{T_i(s^v+\lambda)^3}}{\frac{s^v}{(s^v+\lambda)^3}}=\frac{\sum_{i=1}^{\infty}c_i\varphi_i(s)}{\sum_{i=1}^{\infty}c_i'\varphi_i(s)} \quad (38)$$

where

$$c_3=\frac{k_c(T_d T_i\lambda^2-T_i\lambda+1)}{s_{33}T_i},\quad c_2=\frac{-(k_c(2\lambda\tau_d-1)+s_{32}c_3)}{s_{22}},\quad c_1=\frac{k_c T_d-s_{31}c_3-s_{21}c_2}{s_{11}},\ c_3'=\frac{-\lambda}{s_{33}},\quad c_2'=\frac{s_{32}\lambda+s_{33}}{s_{22}s_{33}},\ c_1'=\frac{-(s_{21}c_2'+s_{31}c_3')}{s_{11}}. \quad (39)$$

The relation (32) could be utilized in the FOPID case, too. Thus, the following FOPID controller parameters could be obtained according to relations (22) and (39)

$$k_c=2s_{11}\lambda c_1+(s_{22}+2\lambda s_{21})c_2+(s_{32}+2\lambda s_{31})c_3$$
$$T_d=\frac{s_{11}c_1+s_{21}c_2+s_{31}c_3}{k_c} \quad (40)$$
$$T_i=\frac{k_c}{\lambda^2 s_{11}c_1+(\lambda s_{22}+s_{21}\lambda^2)c_2+(s_{33}+\lambda s_{32}+s_{31}\lambda^2)c_3}$$

where

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}X_1 & X_2 & X_3\\Y_1 & Y_2 & Y_3\\Z_1 & Z_2 & Z_3\end{bmatrix}^{-1}\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix}. \quad (41)$$

$$X_i=\sum_{j=1}^{3}a_{ij1}u_j,\ Y_i=\sum_{j=1}^{3}a_{ij2}u_j,\ Z_i=\sum_{j=1}^{3}a_{ij3}u_j\quad i=1,2,3. \quad (42)$$

$$u_i=\sum_{k=1}^{3}\sum_{j=1}^{3}g_j l_k' a_{jki},\quad q_i=\sum_{k=1}^{3}\sum_{j=1}^{3}l_j c_k' a_{jki},\quad i=1,2,3. \quad (43)$$

4.3. Optimum pole location selection of fractional order Laguerre basis functions

The designed FOPI and FOPID controllers have three adjustable parameters: η,ω_n,λ. These parameters should be selected to achieve the closed loop system stability. Moreover, the best transient response performance in the presence of control signal constraints should be achieved. This means that these parameters are the solution of the following constrained optimization problem

$$\min:\ J(\lambda,\eta,\omega_n)=\frac{\int_0^T (y(t)-y_d(t))^2 dt}{\int_0^T y_d^2(t)dt}$$
$$s.t:\ u^-<u(t)<u^+ \quad (44)$$

where $y(t)$ is the closed loop system step response, $y_d(t)$ is the desired closed loop system step response, $u(t)$ is the control signal, u^+ and u^- are its upper and lower bounds.

5. Simulation results

To show the performance of the FOPI and FOPID controllers a numerical example is provided. The MATLAB software FMINCON function is employed to solve the optimization problem (44).

Example 2: Cconsider the following fractional order plant

$$G(s)=\frac{0.5}{1.5s^{1.3}+1}\,. \qquad (45)$$

Considering $u^{+}=6, u^{-}=0, \eta=1.3, \omega_n=7.56,$, the optimum value of the Laguerre basis function for FOPI controller is obtained as $\lambda=10.896$ .The resultant FOPI controller is

$$C(s)=5.4095(1+\frac{1}{1.6353s^{1.3}})\,. \qquad (46)$$

For the FOPID controller, $\lambda=3.7898$ is obtained. This leads to the following FOPID transfer function

$$C(s)=7.1193(1+\frac{1}{1.6872s^{1.3}}-0.0921s^{1.3})\,. \qquad (47)$$

Fig.1 compares the closed loop system unit step response obtained from the FOPI and FOPID controllers with the desired one. The FOPID shows superior performance comparing with the FOPI controller. Moreover, the control signal constraints are fulfilled.

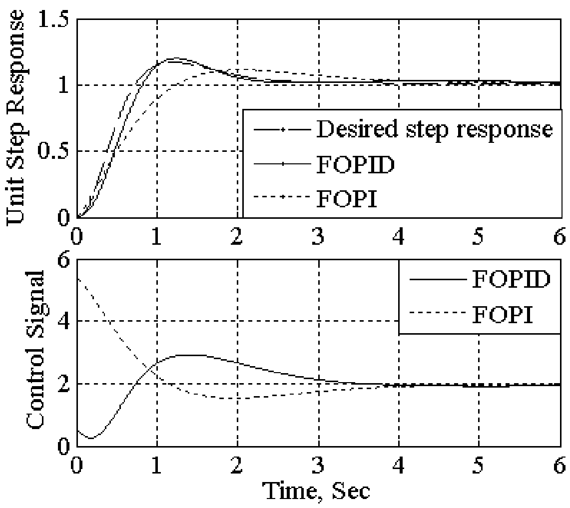

Fig. 1. The closed loop system unit step responses and control signals for Example 2

6. Conclusions

The orthonormal Laguerre basis functions obtained from a Gram-Schmidt orthogonalization approach are employed to design FOPI and FOPID controllers for commensurate order fractional systems. The simulation result shows the effectiveness of the proposed controllers. The best transient response quality based on integral square error performance index in the presence of the control signal limitations is achieved. The design approach could be utilized for stable plants with real poles. Extending the proposed FOPI and FOPID methods for general commensurate order fractional systems could be considered as a future research topic. Designing Laguerre based FOPID controllers for the commensurate order fractional plants with commensurate order smaller than half is another future work. The FOPD controller could be designed in the similar manner, too.

7. References

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