Optimal Sample-Data Output Regulation Based On Realizable Reconstruction Filter

,Saad Ullah#1Muwahida Liquat,
Department of Electrical Engineering, College of Electrical and Mechanical Engineering
National University of Science and Technology
Islamabad,Pakistan
[1]saadullah71@.ee.ceme.edu.pk

Abstract— **An optimal sampled-data output regulation for a class of linear time invariant systems is considered. The essential constraint is the availability of the output samples for the measurement. The main purpose is to explore the response of the linear time invariant systems by introducing a realizable reconstruction filter as digital to analog convertor as compared to the general approach which is zero order hold devices. The gains of the control law which ensures the output regulation has been calculated by introducing the linear quadratic gaussian technique. The proposed scheme ensures the optimal regulation (tracking) in the presence of the external disturbance under the influence that the complete knowledge of the disturbance is known. The corresponding linear time invariant system and the realizable reconstruction filter, whole system will behave like an impulsive system.**

***Keywords*— linear systems, output regulation, linear quadratic gaussian, impulsive systems, sampled-data control.**

I. Introduction

Output regulation is a fundamental problem which has been studied extensively in the control theory since the seminal work done by the Francis in 1977[1], and now we are able to address the problem with the quite complete solution. It addresses the problem by designing a controller which is able to track a class of reference signals and also able to reject a class of disturbances while maintaining the close loop stability of the system. The behaviour of the Linear Time Invariant systems is well understood ([2], [3]). The theory of output regulation is also extended to the Time varying systems in which we solved the differential regulator equations rather than the classical (algebraic) regulator equations ([4], [5]).For sampled-data systems with zero order hold, the output regulation problem is easily solved([6],[7]). But for the general exogenous signals ripples between the sampling periods complicates the matter and are necessary to attenuate for which the continuous time pre compensators are considered [6].

The LQG technique which is composed of optimal state estimation in the presence process and measurement noise followed by state-feedback regulator design using the observed states. The state feedback regulator is designed on the basis of the LQR. Performance of the LQR controller is achieved by selecting the appropriate weighting matrices used in cost function. LQR cost function mostly works by the relation between the states and the input. LQR empower the designer to efficiently use the control effort, which state of the system uses more control effort to get the better response of the respective state, or the transient behaviour of the system. There is a well establish relation between the optimal and robust controllers ([10], [11]).

For the sampled data system there have been two contemporary approaches. In the first scheme we designed the controller and observer such that the continuous-time model of the system is stabilized. Then the designed controller and observer is discretised and implemented. This approach has been implemented in [8] under the assumption that the solution of the continuous-time regulator equation exist. In the second scheme we convert the continuous-time system model in to discrete-time frame of work and then designed the controller and observer in discrete frame of work under the concluded fact that there exists a solution for the discrete time regulator equation, perhaps our overall system converted into discrete domain. The main drawback of this scheme is the failure of the inter-sample behaviour of the controlled continuous system. To cope with this problem is by introducing generalized hold device (GHD) in the controller design.

GHD can be presented as an interpolators with the benefit of the state feedback controllers without using state estimation techniques. By the advantage of using the GHD technique a realizable reconstruction filter is used for the inter-sample behaviour analysis which is presented in [9], [12]. The overall system behaves like an impulsive system in which RRF is designed on the basis of the continuous time internal model of the exosystem and it is also given that the complete knowledge of the exosystem is known. It behaves like an mapping function in-between the discrete input of the controller and the continuous time output to the system just like a digital to analog convertor.

In this way we can use this continuous time input to overcome the intersample behaviour of the system. Thus the overall system performance is better as compared to using a simple zero order hold function.

II. Problem Formulation

Consider the following linear time invariant system that can be represented by the following state space model given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Pw(t) \\ y(t) &= Cx(t) \\ y(k) &= Cx(k) \end{aligned} \qquad (1)$$

Where $x(t) \in R^n$ is the state, $y\epsilon R^P$ is the output and $u\epsilon R^m$ is the input. It is considered that only the sampled measurements of the output are available for measurement. The disturbance present in the system is modelled by$w(t)$.

The model of the exogenous system is given as

$$\dot{w}(t) = \begin{bmatrix} \dot{w_r}(t) \\ \dot{w_d}(t) \end{bmatrix} = \begin{bmatrix} S_r & 0 \\ 0 & S_d \end{bmatrix} w(t)$$
$$y_r(t) = Q_r w_r(t)$$
$$y_r[k] = y_r(k\tau)$$
(2)

Where S_r corresponds to the reference signal which followed by the system output. S_d represents the disturbance which is introduce at the input of the system. According to the theory of the LTI system

III. REALIZABLE RECONSTRUCTION FILTER(RRF)

The input to the LTI system is generated through a GHD known as realizable reconstruction filter (RRF) [9]. The filter is represented as follows

$$\dot{\eta}(t) = A_h \eta(t) \qquad t \neq k\tau$$
$$\eta(k\tau) = \eta(k\tau^-) + B_h u(k) \qquad t = k\tau$$
$$u(t) = C_h \eta(t) \qquad (3)$$

In the above mentioned equations $\eta(t)$ undergoes a jump at the instances of $k\tau$. The time instances used for the realizable reconstruction filter are the integer multiple of the sampling timeτ. The input to the above system is the discrete output of the controller given as $u(k)$ and the output of the impulsive system is the continuous time $u(t)$ to drive the plant. The basic structure of the RRF is based on the dynamics of the exogenous system by taking the assumption as follows

$$A_h = S_r \qquad (4)$$

By constructing the RFF by above assumption has the benefit of the state feedback instead of the state estimation. Therefore the states $\eta(t)$ and its samples $\eta(k)$ are available for measurement .Another advantage of the RRF is the same dimensions of the $u(k)$ and $u(t)$.RRF also ensure the controllability and observability of the complete design.

IV. SAMPLED-DATA OUTPUT REGULATION WITH RFF

The augmented system including LTI in (1) and RRF in (3) behaving like an impulsive system can be written as

$$\dot{x}(t) = \begin{bmatrix} \dot{\eta}(t) \\ \dot{x}(t) \end{bmatrix} = Ax(t) + Bw(t) \qquad t \neq k\tau$$
$$x(k\tau) = \begin{bmatrix} \eta(k\tau) \\ x(k\tau) \end{bmatrix} = x(k\tau^-) + Bu[k] \qquad t = k\tau$$
$$y(t) = Cx(t) \qquad (5)$$

The corresponding matrices are given as

$$\boldsymbol{A} = \begin{bmatrix} A_h & 0 \\ BC_h & A \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} B_h \\ 0 \end{bmatrix}, \boldsymbol{P} = \begin{bmatrix} 0 \\ B * C_h \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 0 & C \end{bmatrix}$$

In the above system the continuous time system undergoes the jump at the integer multiple of τ.

The incorporation of the exogenous system with the above augmented system can be simplifies as given below

$$\dot{\chi}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \boldsymbol{\mathcal{A}}\chi(t) \qquad t \neq k\tau$$
$$\chi(k\tau) = \begin{bmatrix} x(k\tau) \\ w(k\tau) \end{bmatrix} = \chi(k\tau^-) + \mathcal{B}u[k] \qquad t = k\tau$$
$$\Upsilon(t) = C\chi(t) \qquad (6)$$

The corresponding values of the above matrices are

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \boldsymbol{A} & \mathbf{P} \\ 0 & \boldsymbol{S} \end{bmatrix}, \quad \boldsymbol{\mathcal{B}} = \begin{bmatrix} \boldsymbol{B} \\ 0 \end{bmatrix}, C = \begin{bmatrix} \mathbf{C} & Q \end{bmatrix}$$

The state response of the system (6) can be represented as for time interval $t \in [T_k, T_{k+1}]$ can represented as

$$\chi(t) = \Phi(t, t_0)\chi(t_0) + \sum_{j=1}^{k} \Phi(t, T_j)\, \mathcal{B}u[j] \qquad (7)$$

The first part of the equation represents the zero input and second part represents the zero state response of the system. For the output regulation of the impulsive system in (6) the following assumption can be made

A1.

(A_d, B_d) is stabilizable.

A2.

The pair $(C, e^{\mathcal{A}\tau})$ is detectable

A3.

There exist matrices Π(PI) and Γ(GAMMA) which fulfil the regulator equations such that:

$$\Pi S_d = A_d \Pi + B_d + B_d$$
$$0 = C_d \Pi + Q_d \qquad (8)$$

Where

$$A_d = e^{A\tau}, B_d = A_d B, P_d = A_d P, C_d = C$$

The values of the S_d and Q_d corresponds to the discrete time exosystem.In this paper we presented a discrete controller which is followed by a discrete observer for the output regulation of the impulsive system. The designed observer has the property of the stable closed loop asymptotically stability. For the design of the observer the following matrices are used given as

$$\mathbb{A} = \begin{bmatrix} A & P \\ 0 & S_d \end{bmatrix} \quad \mathbb{B} = \begin{bmatrix} BC_h \\ 0 \end{bmatrix} \mathbb{C} = \begin{bmatrix} C & 0 \end{bmatrix} \qquad (9)$$

Above matrices has been used for the state estimation of the system augmented with the exogenous system. Then the discrete observer and controller equations are given as follows

$$\hat{\chi}_0[k+1] = e^{\mathbb{A}\tau}\hat{\chi}_o + e^{\mathbb{A}\tau}\mathbb{B}u[k] + H(y[k] - \mathbb{C}\hat{\chi}_0) \qquad (10)$$
$$u[k] = \begin{bmatrix} F & \Gamma - F\Pi \end{bmatrix} \begin{bmatrix} \eta \\ \hat{\chi}_0 \\ \hat{w} \end{bmatrix} [k] \qquad (11)$$

To solve the tracking problem it remains to design a feedback gain F and the observer gain K. In this paper we have designed the system gains on the basis of the optimal technique Linear, Quadratic, Gaussian (LQG). In short the optimal design process has following steps

(a) Design an optimal regulator for the linear plant assuming full-state feedback and also defining a quadratic objective function given as

$$J(u) = \sum_{k=1}^{\infty} x^T(k)Q(k)x(k) + u^T(k)Ru(k) \quad (12)$$

For the minimization of the above cost function the gain F has been calculated as

$$F = ((B^T M_o B + R)^{-1} B^T M_o A) \quad (13)$$

By solving the regulator equation given as

$$0 = M_0 - A^T M_0 A - Q + A^T M_0 B(R + B^T M_0 B)^{-1} B^T M A - Q \quad (14)$$

(b) Design a kalman filter for the plant assuming a control input u (t) a measured output y (t) with the known noise covariance. All these noises are zero mean Gaussian and statistically independent explained as follows

$$E[\xi(t)] = E[\vartheta(t)] = 0 \qquad E[\xi(t)\xi(t)^T] = \Xi \geq 0$$

$$E[\vartheta(t)\vartheta(t)^T] = \Theta > 0 \qquad E[\xi(t)\vartheta^T] = 0$$

For this paper we have selected the process and measurement noise covariance's which are $\Xi = [10^{-3}]$ and $\Theta = [10^{-7}]$. And the gain is calculated by as follows

$$H = \mathbb{R}_o \mathbb{C}^T(\mathbb{C}\mathbb{R}_o\mathbb{C}^T + \mathbb{R}_o)^{-1}$$

The above gain has been calculated by solving the following discrete time filtering equation

$$S = \mathbb{A}S\mathbb{A}^T + \mathbb{Q}_o \quad - \mathbb{A}S\mathbb{C}^T(\mathbb{A}S\mathbb{A}^T + \mathbb{R}_o)^{-1}\mathbb{C}S\mathbb{A} \quad (15)$$

The above calculated gains fulfils the output regulation for the sample-data system where F is chosen on the basis of LQR technique, such that the pair $(\boldsymbol{A}_d + \boldsymbol{B}_d F)$ fall within the unit circle. The gain (H) that has been calculated for the output feed-back system on the basis of the optimal state estimator (LQG) such that the pair $(e^{\mathbb{A}\tau} - H\mathbb{C})$ is exponentially stable. The LQG controller corresponds to the combination of the kalman filter equation and the LQR.

V. EXAMPLE

The above designed technique has been implemented to an inverted pendulum with cart. By taking a cart with an inverted pendulum hinged on the top of it as shown in the fig.1 for the simplicity we assumed that the cart and pendulum moves in one direction. The friction and the mass of the stick is neglected. The aim is to maintain the pendulum in its vertical position. This simple mechanism can be used as a space booster on its take off position.

After the linearization of the system we get the following differential equations.

$$(M + m)\ddot{y} + ml\ddot{\theta} = u \quad (16)$$

$$2l\ddot{\theta} - 2g\theta + \ddot{y} = 0 \quad (17)$$

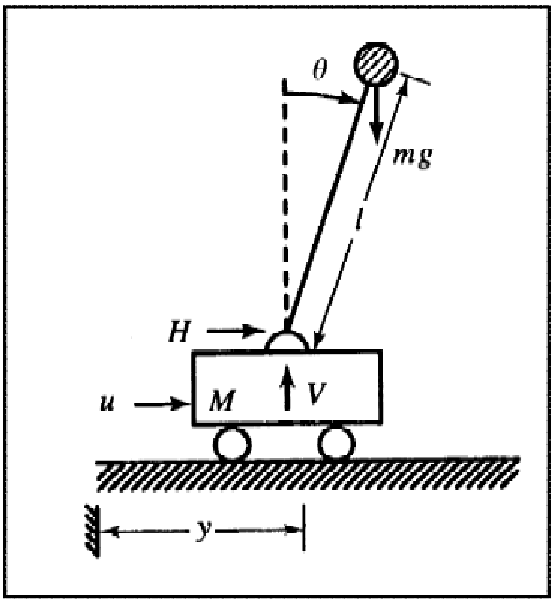

Fig.1 Inverted Pendulum

The state space representation of the system can be formed as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-2mg}{2M+m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{2g(M+m)}{(2M+m)l} \end{bmatrix}, C = [1 \quad 0 \quad 0 \quad 0]$$

$$B = \begin{bmatrix} 0 \\ \frac{2}{2M+m} \\ 0 \\ \frac{-1}{(2M+m)} \end{bmatrix} \quad (18)$$

For the simplification of the model we took the following assumptions

$$\frac{-2mg}{2M+m} = 1, \ \frac{2g(M+m)}{(2\ M+m)l} = 5, \ \frac{2}{2M+m} = 1, \ \frac{-1}{(2M+m)} = 2$$

The model is designed under simplification and also linearized so it is only applicable to small change in θ and $\dot{\theta}$.

REGULATION OF A CONSTANT SIGNAL

The linearized system represented in (13, 14), we want it to track the signal that has been generated by the system in (3), can be written as:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$\dot{w}(t) = 0_{2x1} \quad (19)$$
$$y(t) = Cx(t)$$

For the system represented in (19), we choose the sampling time $\tau = 0.1 sec$ and also assumed that the control input is realized through realizable reconstruction filter and all respective measurements are taken at the fixed sampling instant $k\tau$. The value of F is calculated by choosing the appropriate value of Q and R which is for this case is Q = diag {1, 100, 10, 10,1}, R= {0.1} .

The value of the observer represented in (14) has been calculated by choosing $\mathbb{Q}_o$ =diag{1,1,1,10,10} and $\mathbb{R}_o$={0.001}.
The following result has been shown the comparison between the zero order hold response and the output of the system with RRF when there is no constant disturbance and noise is introduced in the system

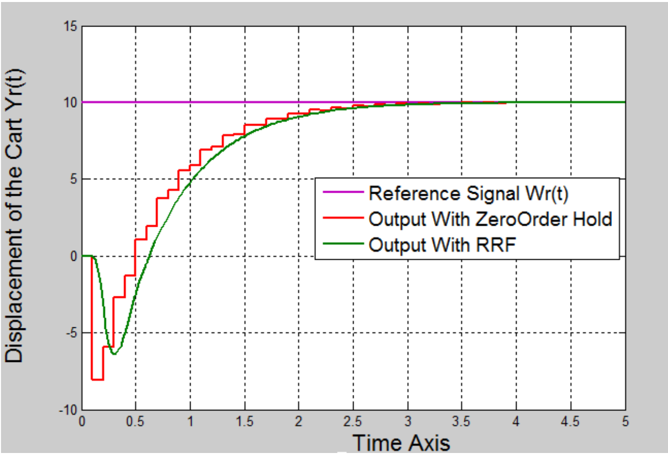

Fig (2) output comparison of the system

The error of the system in comparison with the zero order hold and RRF is presented as follows

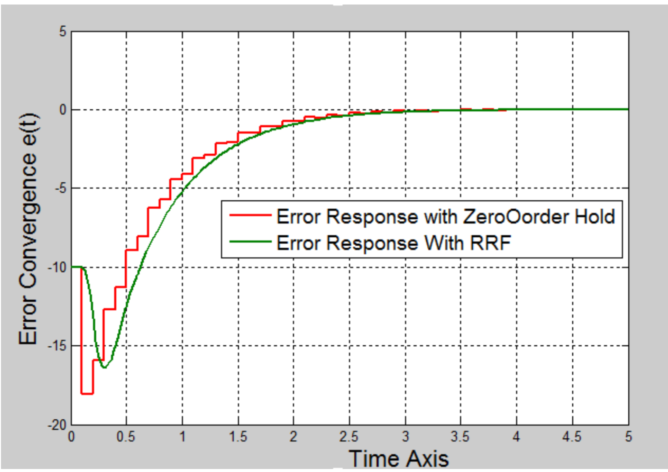

Fig (3) Error Plot

The comparison between the zero order hold and RRF with respect to their control effort has also shown in the fig (4).

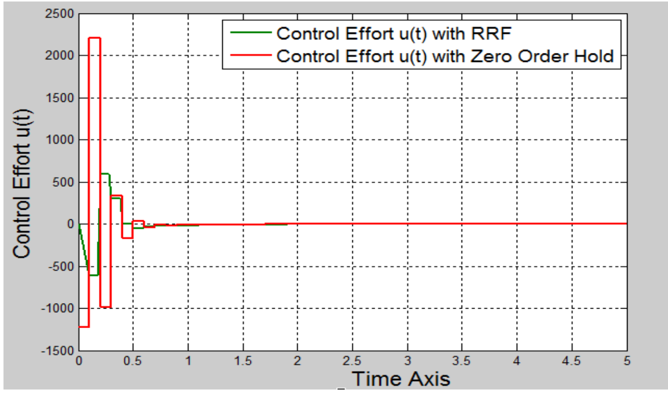

Fig (4) control effort comparison u(t)

The results shown in the following figures represents the output of the system when there has been a constant disturbance and white Gaussian noise introduced in the system.
The constant disturbance is considered as the same the reference signal and on the other hand we have selected the process and measurement noise covariance's which are $\Xi = [10^{-3}]$ and$\Theta = [10^{-7}]$.

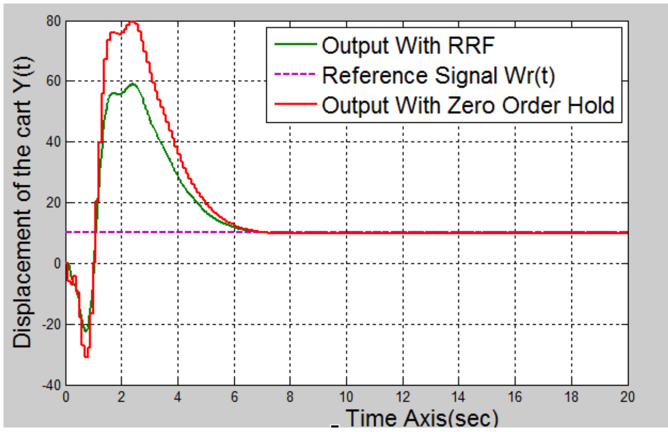

Fig (5) Displacement of the cart with perturbations

The error convergence of the system is shown as follows in the fig (6).

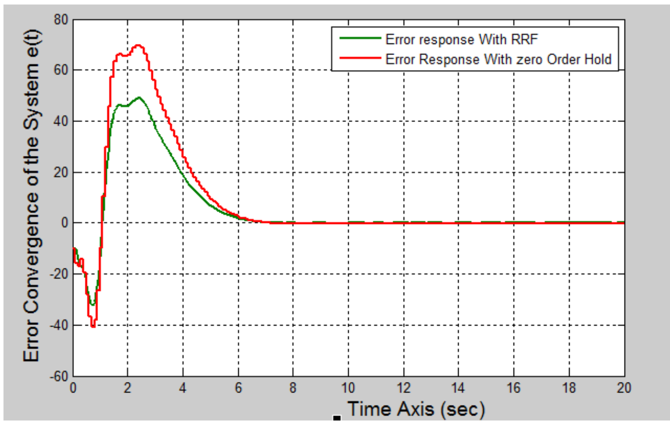

Fig (6) Error convergence with noise

The control effort that has been utilized by the system in the presence of perturbation, comparison with the zero order hold and RRR is also shown in the fig (7).

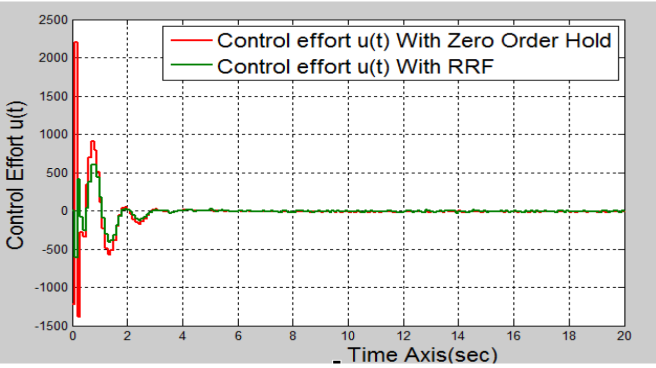

Fig (7) control effort u (t)

CONCLUSION

The optimal sampled-data output regulation is presented for a class of linear time invariant system is considered and realizable reconstruction filter has been used as an analog to digital convertor as compared to general approach of using zero order hold. By the use of realizable reconstruction filter with the linear time invariant plant are modelled as linear impulsive system. Since the realizable reconstruction filter contains the dynamics of the reference signal that can be used as a control variable to overcome the intersample behaviour of the closed loop system which effects system dynamic change at the sampling instants which alternatively give advantage in the regulation and control energy efficiency as depicted in the simulations. In the above scheme there has been a constant disturbance which is generated by the exogenous system and the process noise, measurement noise is also handled through the LQG technique. The above proposed technique has been implemented on an inverted pendulum with cart and results are shown through simulations.

REFERENCES

[1] B. A. Francis, "The linear multivariable regulator problem," *SIAM J. Contr. and Opt.*, vol. 15, pp. 486-505, 1977.

[2] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Auotmatica*, vol. 12, pp. 457-465, 1976.

[3] A. Saberi, A. A. Stoorvogel, and P. Sannuti, "Control of linear systems with regulation and input constraints," Springer, 2000.

[4] A. Ichikawa and H. Katayama, "Output regulation of periodic systems," *Proceedings of the IFAC Workshop on Periodic Control Systems,* Yokohama, Japan, 2004, pp. 225-230.

[5] A. Ichikawa and H. Katayama, "Output regulation of descriptor systems with periodic coefficients," *Proceeding of 1the Joint 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC'05)*, Seville, Spain 2005, pp. 3339-3344.

[6] Y. Yamamoto, "A function space approach to sampled-data control systems and tracking problems," *IEEE Trans*. Automat. Contr., vol. 39, 1994, pp. 703-713.

[7] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*, Springer, 1995

[8] H.Katayama and A.Ichikawa."output regulation for Sampled-data systems with application to marine sysyems" proceedings of 46th IEEEConference on Decision and control(CD07),pp.1058-1063,New Orleans,USA,2007

[9] Muwahida Liaquat and Mohammad Bilal Malik, "Discrete Time Output Regulation on Sample-Data Systems", International National Journal of Computer Theory and Engineering,vol 4,No 5,2012

[10] F.lin,"An Optimal Control Approach to Robust Control Design", International journal of control, vol 73(3),pp.177-186,2000

[11] Willems and Mareels," A Rigorous Solution of the infinite Time Interval LQ Problem With Constant State Tracking:, System And Control letters vol,52,pp,289-296,2004.

[12] Muwahida Liaquat and Mohammad Bilal Malik " Sample-Data Regulation of a Class of Time Varying Systems Based on a Realizable Reconstruction Filter", IEEE International Conference on Smart Instrumentation ,Measurement and Applications (ICSIMA) National Journal of Computer Theory and Engineering,vol 4,No 5,2012