Bandwidth Efficient Maximum Likelihood SNR Estimators for QAM Signals in Complex AWGN Channel

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Abstract

The signal-to-noise ratio (SNR) is often unknown to the receiver in most wireless communication systems. In order to perform various functions, SNR needs to be estimated. This paper deals with the estimation of SNR in a system that is based on quadrature amplitude modulation (QAM) signals transmitted through a complex additive white Gaussian noise (AWGN) channel. The estimator is designed for non-data-aided (NDA) and partially data-aided (PDA) scenarios using the maximum likelihood approach. The Cramer-Rao lower bound (CRLB) is also derived for the estimators. Different types of square and cross QAM have been used for performance evaluation of the NDA estimator. The PDA case has been observed for different ratios of pilot and data symbols. Performance of estimators has been compared with CRLB for all cases.

1. Introduction

In many wireless communication systems, the value of signal-to-noise ratio (SNR) is unknown to the receiver. An estimate of SNR is required, which can be used to perform different operations at the receiver end. Some applications include handoffs in mobile communication, power control, turbo decoding, etc. Many authors have estimated SNR for different types of signals using various techniques, including maximum likelihood (ML), method of moments (MM), and decision-directed (DD) method, etc. Maximum likelihood estimation of SNR has been presented for frequency shift keying (FSK) in [1, 2]. References [3-5] deal with maximum likelihood SNR estimation for phase shift keying (PSK) signals. The problem of SNR estimation for quadrature amplitude modulation (QAM) has been proposed in [6, 7] using the method of moments. The ML estimation of signal-to-interference-noise ratio (SINR) for QAM signals has been discussed in [8].

In this paper, we have proposed a maximum likelihood (ML) SNR estimator for different types of QAM constellations, i.e. square and cross QAM in complex additive white Gaussian noise (AWGN) channel. The data used for this estimator may be pilot symbols or data symbols. When only pilot symbols are used for estimation, the method is called data-aided (DA) estimation. When data symbols are used for estimation the estimation technique is termed as non-data-aided (NDA) estimation. Partially data-aided (PDA) estimation can also be done when a combination of pilot and data symbols is used. We have presented the NDA and PDA SNR estimation for QAM signals in this paper. Cramer-Rao lower bound (CRLB) has also been derived for the estimator, which is a lower bound on the variance of an unbiased estimator.

The rest of the paper has been organized such that system model has been presented in Section 2. Section 3 deals with the maximum likelihood SNR estimator design for PDA and NDA cases. The CRLB has been derived in Section 4. Simulation results are discussed in Section 5 and the discussion is concluded in Section 6.

2. System Model

Consider a wireless communication system where QAM signals are passed through complex AWGN channel. The output of the receiver's matched filter is the signal of interest for estimating SNR. This signal can be represented as:

$$r_i = \sqrt{S}m_i + \sqrt{N}n_i \tag{1}$$

where r_i, m_i and n_i are the i^{th} samples of received signal, transmitted message signal and complex AWGN respectively. $\sqrt{S}$ and $\sqrt{N}$ represent the signal and noise power scale factors, respectively, taken in square root form to obtain a simplified estimator expression. The received signal and its components are complex, so it can be written as:

$$r_i = r_{i_I} + jr_{i_Q} \tag{2}$$

where I and Q represent the in-phase and quadrature components of the received signal. These in-phase and quadrature components can be given as:

$$r_{i_I} = \sqrt{S}m_{i_I} + \sqrt{N}n_{i_I} \tag{3}$$

$$r_{i_Q} = \sqrt{S}m_{i_Q} + \sqrt{N}n_{i_Q} \tag{4}$$

The parameters to be estimated are S and N, as the SNR is given as:

$$\gamma = \frac{S}{N} \tag{5}$$

The data packet used for estimating SNR is of length g, such that the observed received signal packet is $\boldsymbol{r} = [r_1\ r_2\ \dots\ r_g]$. We assume that the number of pilot symbols and data symbols is k and l respectively. The total packet length is thus $g = k + l$.

3. Maximum Likelihood Estimation

In order to find the maximum likelihood estimate of SNR, the values of S and N have to be estimated using the ML approach. The ratio of these estimated values gives us the estimate of SNR.

$$\hat{\gamma}_{ML} = \frac{\hat{S}_{ML}}{\hat{N}_{ML}} \tag{6}$$

where $\hat{\gamma}_{ML}$, $\hat{S}_{ML}$ and $\hat{N}_{ML}$ are maximum likelihood estimates of the SNR, signal power S and noise power N respectively. The joint probability density function (PDF) of in-phase and quadrature components of AWGN can be written as:

$$p\left(n_{i_I}, n_{i_Q}\right) = \frac{1}{\pi N} \exp\left(\frac{-({n_{i_I}}^2 + {n_{i_Q}}^2)}{N}\right) \tag{7}$$

Thus, the joint PDF of the received symbol r_i can be written as:

$$\begin{aligned} p(r_i) &= p\left(r_{i_I}, r_{i_Q}\right) \\ &= \frac{1}{\pi N} \exp\left(-\frac{\left(r_{i_I} - \sqrt{S} m_{i_I}\right)^2 + \left(r_{i_Q} - \sqrt{S} m_{i_Q}\right)^2}{N}\right) \end{aligned} \tag{8}$$

3.1. NDA ML Estimator

In non-data aided (NDA) estimation of SNR, we estimate SNR using actual received data without the knowledge of transmitted data. In this case, the received data is demodulated and detected, and this estimated message signal is used to estimate the SNR. This method requires no additional bandwidth as there is no pilot symbol transmission. For the NDA case, the whole data packet consists of l data symbols, so we have $g = l$. The joint PDF presented in the previous section can be re-written for $\boldsymbol{r}$, i.e. l data symbols as:

$$\begin{aligned} p(\boldsymbol{r_I}, \boldsymbol{r_Q}) &= \prod_{i=1}^{l} p\left(r_{i_I}, r_{i_Q}\right) \\ &= \frac{1}{(\pi N)^l} \exp\left[\frac{-1}{N}\left(\sum_{i=1}^{l}\left(r_{i_I} - \sqrt{S}\hat{m}_{i_I}\right)^2 + \sum_{i=1}^{l}\left(r_{i_Q} - \sqrt{S}\hat{m}_{i_Q}\right)^2\right)\right] \end{aligned} \tag{9}$$

where $\boldsymbol{r_I}$ and $\boldsymbol{r_Q}$ represent the in-phase and quadrature components of the received data packet $\boldsymbol{r}$. Here m_{i_I} and m_{i_Q} are not known, so we have used their estimates found by the receiver's detection module, written as $\hat{m}_{i_I}$ and $\hat{m}_{i_Q}$. The log-likelihood function can be written as:

$$\begin{aligned} \Lambda_{NDA} &= \ln p(\boldsymbol{r_I}, \boldsymbol{r_Q}) \\ &= -l \ln(\pi N) - \frac{1}{N}\left[\sum_{i=1}^{l}\left(r_{i_I} - \sqrt{S}\hat{m}_{i_I}\right)^2 + \sum_{i=1}^{l}\left(r_{i_Q} - \sqrt{S}\hat{m}_{i_Q}\right)^2\right] \end{aligned} \tag{10}$$

In order to find the required ML estimates, i.e. $\hat{S}_{ML}$ and $\hat{N}_{ML}$, we differentiate (10) with respect to S and N respectively, to get:

$$\hat{S}_{ML} = \left[\frac{\frac{1}{l}\sum_{i=1}^{l}\left(r_{i_I}\hat{m}_{i_I} + r_{i_Q}\hat{m}_{i_Q}\right)}{\frac{1}{l}\sum_{i=1}^{l}\left[\left(\hat{m}_{i_I}\right)^2 + \left(\hat{m}_{i_Q}\right)^2\right]}\right]^2 \tag{11}$$

and

$$\hat{N}_{ML} = \frac{1}{l}\sum_{i=1}^{l}\left[\left(r_{i_I}\right)^2 + \left(r_{i_Q}\right)^2\right] - \hat{S}_{ML}\frac{1}{l}\sum_{i=1}^{l}\left[\left(\hat{m}_{i_I}\right)^2 + \left(\hat{m}_{i_Q}\right)^2\right] \tag{12}$$

The average energy of a signal is calculated as:

$$\varepsilon_{av} = \frac{1}{l}\sum_{i=1}^{l}\left[\left(m_{i_I}\right)^2 + \left(m_{i_Q}\right)^2\right] \tag{13}$$

For QAM constellation, the average energy has been given in [9]. For square M-QAM:

$$\varepsilon_{av} = d^2\left(\frac{M-1}{6}\right) \tag{14}$$

For cross M-QAM:

$$\varepsilon_{av} = \frac{d^2}{6}\left(\frac{31}{32}M - 1\right) \tag{15}$$

where M is the number of constellation points and d is the distance between nearest neighbors within the constellation.

It can be seen in (11) and (12) that the average energy term is present, so these can be simplified as:

$$\hat{S}_{ML} = \frac{\left[\frac{1}{l}\sum_{i=1}^{l}\left(r_{i_I}\hat{m}_{i_I} + r_{i_Q}\hat{m}_{i_Q}\right)\right]^2}{\varepsilon_{av}^2} \tag{16}$$

and

$$\hat{N}_{ML} = \frac{1}{l}\sum_{i=1}^{l}\left[\left(r_{i_I}\right)^2 + \left(r_{i_Q}\right)^2\right] - \hat{S}_{ML}\varepsilon_{av} \tag{17}$$

As discussed in Section 2, the assumption of having signal and noise power in square root form is advantageous. Otherwise, the expressions obtained above would represent estimates of squares of S and N, thus increasing computational complexity. Using (16) and (17), the estimated SNR for NDA case can be written as:

$$\begin{aligned} \hat{\gamma}_{ML} &= \frac{\hat{S}_{ML}}{\hat{N}_{ML}} \\ &= \frac{\left[\frac{1}{l}\sum_{i=1}^{l}\left(r_{i_I}\hat{m}_{i_I} + r_{i_Q}\hat{m}_{i_Q}\right)\right]^2}{{\varepsilon_{av}}^2\frac{1}{l}\sum_{i=1}^{l}|r_i|^2 - \varepsilon_{av}\left[\frac{1}{l}\sum_{i=1}^{l}\left(r_{i_I}\hat{m}_{i_I} + r_{i_Q}\hat{m}_{i_Q}\right)\right]^2} \end{aligned} \tag{18}$$

This is the final expression of estimator for NDA case. It can be seen that it does not depend on the transmitted symbols, rather it depends on the estimates of transmitted symbols obtained from the received symbols by the detector.

3.2. PDA ML Estimator

The ML estimator for partially data aided (PDA) case has been derived in this subsection. In this case the received data packet used is comprised of pilot and data symbols. According to the system model described in Section 2, there are k pilot symbols and l data symbols, making the total packet length $g = k + l$. This estimator requires more bandwidth than the NDA case but less bandwidth than the DA case. The performance of this estimator should also be better than the NDA estimator as some symbols are known to the receiver. The performance was found to improve with the increase in number of pilot symbols as compared to data symbols.

In order to derive the PDA ML estimator, we first require the joint PDFs of the received pilot and data symbols. The joint PDF for pilot symbols is given by (8) and for data symbols it can be written as:

$$p(r_i) = p\left(r_{i_I}, r_{i_Q}\right) = \frac{1}{\pi N}\exp\left(-\frac{\left(r_{i_I} - \sqrt{S}\hat{m}_{i_I}\right)^2 + \left(r_{i_Q} - \sqrt{S}\hat{m}_{i_Q}\right)^2}{N}\right) \tag{19}$$

The joint PDF for $\boldsymbol{r}$, i.e. the complete received packet can be written as:

$$\begin{aligned} p(\boldsymbol{r_I}, \boldsymbol{r_Q}) &= \prod_{i=1}^{k} p\left(r_{i_I}, r_{i_Q}\right)\prod_{i=1}^{l} p\left(r_{i_I}, r_{i_Q}\right) \\ &= \frac{1}{(\pi N)^{k+l}}\exp\left[\frac{-1}{N}\left(\sum_{i=1}^{k}\left(r_{i_I} - \sqrt{S}m_{i_I}\right)^2 + \sum_{i=1}^{k}\left(r_{i_Q} - \sqrt{S}m_{i_Q}\right)^2 + \sum_{i=1}^{l}\left(r_{i_I} - \sqrt{S}\hat{m}_{i_I}\right)^2 + \sum_{i=1}^{l}\left(r_{i_Q} - \sqrt{S}\hat{m}_{i_Q}\right)^2\right)\right] \end{aligned} \tag{20}$$

It can be seen that the log-likelihood function found from the joint PDF of (20) is similar to (10), except for the splitting of data packet into two portions. So, the maximization of the log-likelihood function for PDA case is done in the same way as in Section 3.1, yielding the estimator equations for S and N as:

$$\hat{S}_{ML} = \left[\frac{\frac{1}{k}\sum_{i=1}^{k}\left(r_{i_I}m_{i_I} + r_{i_Q}m_{i_Q}\right) + \frac{1}{l}\sum_{i=1}^{l}\left(r_{i_I}\hat{m}_{i_I} + r_{i_Q}\hat{m}_{i_Q}\right)}{\frac{1}{k}\sum_{i=1}^{k}\left[\left(m_{i_I}\right)^2 + \left(m_{i_Q}\right)^2\right] + \frac{1}{l}\sum_{i=1}^{l}\left[\left(\hat{m}_{i_I}\right)^2 + \left(\hat{m}_{i_Q}\right)^2\right]}\right]^2 \tag{21}$$

and

$$\begin{aligned} \hat{N}_{ML} = {} & \frac{1}{k}\sum_{i=1}^{k}\left[\left(r_{i_I}\right)^2 + \left(r_{i_Q}\right)^2\right] + \frac{1}{l}\sum_{i=1}^{l}\left[\left(r_{i_I}\right)^2 + \left(r_{i_Q}\right)^2\right] \\ & - \hat{S}_{ML}\left[\frac{1}{k}\sum_{i=1}^{k}\left[\left(m_{i_I}\right)^2 + \left(m_{i_Q}\right)^2\right] + \frac{1}{l}\sum_{i=1}^{l}\left[\left(\hat{m}_{i_I}\right)^2 + \left(\hat{m}_{i_Q}\right)^2\right]\right] \end{aligned} \tag{22}$$

By using the average energy equations for square and cross QAM given in (14) and (15), we can simplify the estimator expression to get:

$$\hat{\gamma}_{ML} = \frac{\hat{S}_{ML}}{\hat{N}_{ML}} = \frac{\left[\frac{1}{k}\sum_{i=1}^{k}\left(r_{i_I}m_{i_I} + r_{i_Q}m_{i_Q}\right) + \frac{1}{l}\sum_{i=1}^{l}\left(r_{i_I}\hat{m}_{i_I} + r_{i_Q}\hat{m}_{i_Q}\right)\right]^2}{\left[\begin{array}{c}\varepsilon_{av}^2\left[\frac{1}{k}\sum_{i=1}^{k}|r_i|^2 + \frac{1}{l}\sum_{i=1}^{l}|r_i|^2\right] - \\ \varepsilon_{av}\left[\frac{1}{k}\sum_{i=1}^{k}\left(r_{i_I}m_{i_I} + r_{i_Q}m_{i_Q}\right) + \frac{1}{l}\sum_{i=1}^{l}\left(r_{i_I}\hat{m}_{i_I} + r_{i_Q}\hat{m}_{i_Q}\right)\right]^2\end{array}\right]} \tag{23}$$

The performance of this estimator is different for different lengths of pilot and data packet, i.e. k and l, while keeping the total packet length, g, same.

4. Cramer-Rao Lower Bounds

The performance of an un-biased estimator can be evaluated by a lower bound on the variance of estimator, known as the Cramer-Rao lower bound (CRLB). In this section we have derived this bound for the designed estimators, considering the fully data-aided scenario. The same bound can be used for both NDA and PDA cases. However, it will be seen that when data symbols are used, the estimator has some bias in the low SNR region, so in this region the estimator performance differ from the bound. The CRLB of an estimator is given by [10] as:

$$CRLB = \frac{\partial f(\theta)}{\partial \theta} I^{-1}(\theta) \frac{\partial f(\theta)}{\partial \theta}^T \tag{24}$$

where,

$$f(\theta) = \frac{S}{N} \tag{25}$$

and

$$\frac{\partial f(\theta)}{\partial \theta} = \begin{bmatrix}\frac{1}{N} & \frac{-S}{N^2}\end{bmatrix} \tag{26}$$

In (24), $I(\theta)$ is called the Fisher Information Matrix, which is given as:

$$I(\theta) = \begin{bmatrix} -E\left[\frac{\partial^2 \Lambda}{\partial S^2}\right] & -E\left[\frac{\partial^2 \Lambda}{\partial S \partial N}\right] \\ -E\left[\frac{\partial^2 \Lambda}{\partial S \partial N}\right] & -E\left[\frac{\partial^2 \Lambda}{\partial N^2}\right] \end{bmatrix} \tag{27}$$

The elements of Fisher Information matrix are calculated using partial derivatives, and the matrix is found to be:

$$I(\theta) = \begin{bmatrix} \frac{g\varepsilon_{av}}{2NS} & 0 \\ 0 & \frac{g}{N^2} \end{bmatrix} \tag{28}$$

which gives the CRLB after putting values in (24) as:

$$var\{\hat{\gamma}\} \geq \frac{2\gamma}{g\varepsilon_{av}} + \frac{\gamma^2}{g} \tag{29}$$

When normalized with respect to γ^2, the asymptotic behavior of the CRLB can be observed with increase in SNR, which becomes normalized mean square error (NMSE) for an unbiased estimator:

$$NMSE\{\hat{\gamma}\} \geq \frac{2}{g\gamma\varepsilon_{av}} + \frac{1}{g} \tag{30}$$

This bound has been used to check the performance of the estimators, unbiased estimators approach CRLB.

5. Performance Evaluation

The simulation results for both estimators have been presented and discussed in this section. The total packet length g has been fixed to 1000 symbols for NDA and PDA cases. The results have been averaged for 10,000 trials for square and cross QAM. The performance has been evaluated on the basis of two parameters, the normalized mean square error (NMSE) and the normalized sample bias. Small NMSE represents good performance of the estimator and the bias has been plotted to see whether the estimator in unbiased over the whole SNR region or if it has bias over some region of SNR. The NMSE of the estimators has been calculated as:

$$NMSE\ \{\hat{\gamma}_{ML}\} = E[(\hat{\gamma} - \gamma)^2] \tag{31}$$

The normalized sample bias has been given in [5] as:

$$\frac{Bias\{\hat{\gamma}_{ML}\}}{\gamma} = \frac{1}{N_t}\sum_{i=1}^{N_t}\frac{\hat{\gamma}_i - \gamma}{\gamma} \tag{32}$$

The performance of NDA and PDA estimators has been discussed in the next subsections.

5.1. NDA Estimator Performance

The NDA estimator uses only data symbols, so the transmitted symbols are not known to the receiver. To cater this, we use the estimates of received symbols found by the detector, which may have error, especially in the low SNR region. Owing to the use of estimated message symbols, the estimator has some bias present in the low SNR region. Therefore, this method reduces bandwidth requirement of the system at the cost of performance degradation. The estimator has been evaluated for various types of square M-QAM, i.e. 4-QAM, 16-QAM, 64-QAM and 256-QAM, and for cross M-QAM, i.e. 32-QAM and 128-QAM on the basis of NMSE and normalized sample bias for a packet length $g = l$, where l is the number of data symbols. As described earlier in Section 5, the packet length has been fixed to 1000 symbols per packet for all types of QAM constellations.

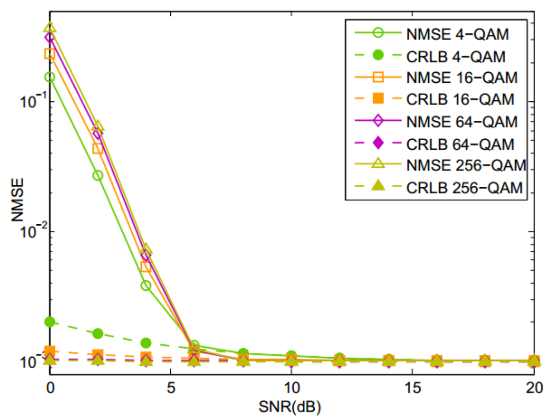

Fig. 1. NMSE and CRLB of SNR estimates for different types of square QAM

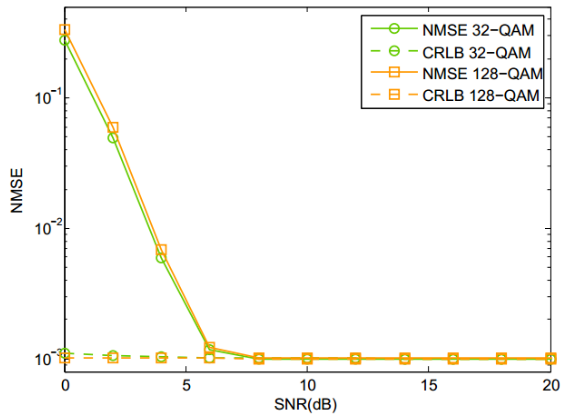

Fig. 2. NMSE and CRLB of SNR estimates for different types of cross QAM

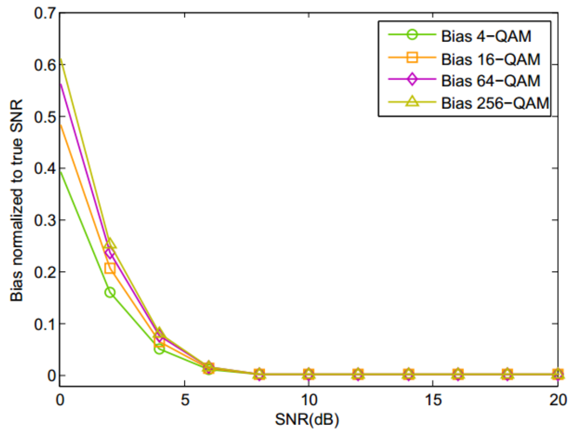

Fig. 3. Bias of the estimator for square M-QAM

The NMSE and CRLB for square QAM have been shown in Fig. 1, while Fig. 2 is for cross QAM. It can be seen that these estimators have some bias in the low SNR region, i.e. when SNR is less than 10 dB. As the true message signal is not known, this method has a greater chance of giving an erroneous estimator. It can be seen that the performance degrades in the low SNR region as we increase the number of constellation points, i.e. M. However, these estimators approach the CRLB with increasing SNR, which is a property of maximum likelihood estimators. The bias for these estimators has been calculated using (32) and plotted in Fig. 3 and Fig. 4 for the different types of square and cross M-QAM respectively. It can also be seen from the plots of bias that the estimator bias increases as we increase the number of constellation points. The degradation in performance in the low SNR region is due to the fact that probability of erroneous detection increases with increase in constellation size and decrease in SNR.

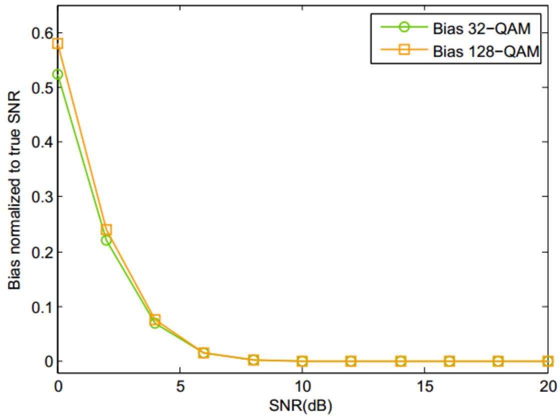

Fig. 4. Bias of the estimator for cross M-QAM

5.2. PDA Estimator Performance

The NMSE and CRLB have been evaluated for PDA using different values of k and l, i.e. number of pilot and data symbols while keeping the total packet length g same, i.e. 1000 symbols. The plots for 32-QAM have been shown in Fig. 5 and Fig. 6. The number of pilot symbols k has been changed from 100 to 200, 300, 400 and 500. It can be seen that as we increase the number of pilot symbols in the packet, the NMSE is reduced in the low SNR region. There is significant difference between NMSE for $k = 100$ and $k = 500$, i.e. the NMSE has been reduced by almost 3 times for $k = 500$.

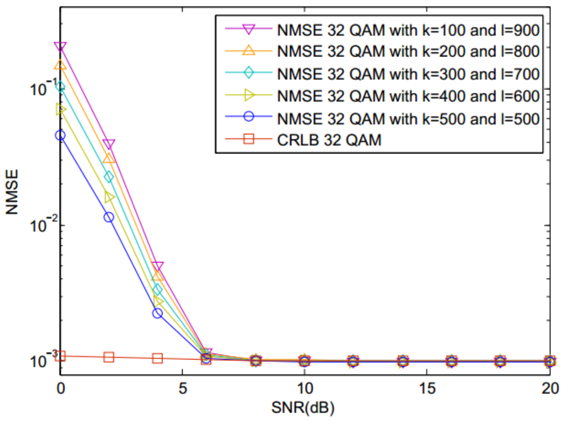

Fig. 5. NMSE and CRLB for 32-QAM with different number of pilot and data symbols

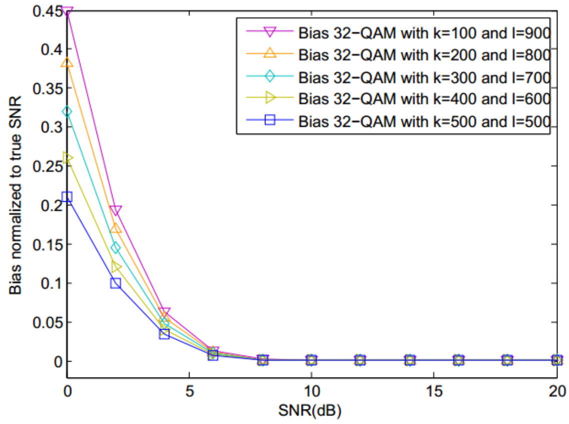

Fig. 6. Bias for 32-QAM with different number of pilot and data symbols

6. Conclusions

The maximum likelihood SNR estimators for square and cross QAM have been designed in complex AWGN using the NDA and PDA approaches. The results show that while NDA estimator has greater bandwidth efficiency, the estimator has some bias in the low SNR region which can be reduced by using the PDA approach. The bias is reduced further as we increase the number of pilot symbols in the PDA case. Both estimators approach CRLB as SNR increases.

7. References

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