# Large-Signal and Small-Signal Analysis of Nonlinear Three-Terminal Switching Elements

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***Abstract*—In this paper, a systematic and efficient method to analyze nonlinear circuits which have three-terminal nonlinear elements like transistors (JFET, MOSFET, BJT) is presented. Large-signal and small-signal analysis of nonlinear circuits containing these elements are achieved. The nonlinear circuits have non-algebraic equations. The solutions of these equations are based on an algorithm called Newton-Raphson algorithm. The method allows computer-aided analysis of nonlinear circuits to be realized efficiently. One illustrative example is included in the study.**

***Keywords:*** *Nonlinear three-terminal switching elements, large-signal, small-signal, analysis, Newton-Raphson algorithm*.

## I. INTRODUCTION

Power electronic systems are widely used in many applications ranging from computing and communications to medical electronics, transportation, industry applications and high-power transmission. These systems generally require switching circuits consisting of semiconductor switches such as transistors and diodes, along with passive elements such as inductances, capacitors and resistors and integrated circuits for control [1].

In practice, while the conventional resistor is probably quite common circuit element, the transistor is certainly the most useful electronic device. A transistor is a three-terminal device behaving as a two-terminal resistor when viewed from any pair of terminals at dc. There are many n-terminal electronic devices available also behave like n-terminal resistors at dc and low-frequency operations [2]. From a different viewpoint, a multi-terminal element is usually modeled as a subcircuit consisting of only two-terminal elements. Thus, a common starting point for studying circuit simulation is to restrict the formulation to two-terminal resistors. In real world, there can be faced with the same situation and it requires dealing with elements such as two-terminal resistors and three-terminal device like transistors. Many methods have been used to analysis of these nonlinear circuits [1-8], [10-14]. Nonlinear electrical properties of some three-terminal switching elements are given in [15-16].

Analysis of circuits including nonlinear elements has two basic problems: (1) obtaining equations of the circuit, (2) solution of the equations with appropriate methods. Nonlinear circuits require intensive mathematical operations and so solving of the operations is mostly difficult. If the analysis doesn't require a solution in a wide range (large-signal analysis), the linearization of nonlinear elements is resorted. For some elements, this linearization process can be done considerably in a wide range in terms of the place of use. As regards some of the others, the linearization can only be valid in a closed spacing depending on the place of use [9].

In this paper, primarily, large signal analysis of nonlinear three-terminal switching devices with Newton-Raphson algorithm is considered and so values of the operating point ($Q$) is obtained. And then, analysis of small-signal equivalent circuit is achieved using the values of this operating point ($Q$).

## II. SMALL-SIGNAL ANALYSIS

Small-signal analysis is an analysis method valid for a nonlinear circuit which is excited by direct and alternating sources while the amplitude of the alternating sources is smaller than which of the direct sources. In other words, while $e(t)<E$, the analysis for the nonlinear circuit seen from Fig.1 is small-signal analysis. Because of frequently encountered with these situation in electronic devices, examination of behaviors in small-signals of nonlinear elements, model extraction of them, and studying of small-signal analysis using these models have gained importance. The behaviors and properties in small-amplitude alternating signals of nonlinear elements have been studied [2], [3], [9].

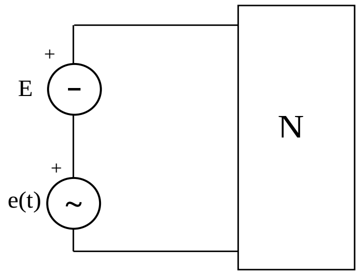


Fig.1. The nonlinear circuit including dc and ac sources

In small-signal analysis of nonlinear circuits, linearization point (operating point-$Q$) is determined by direct sources. After linearization, the circuit is accepted as a linear for small amplitude variable signals around this point. Linearization means that the straight is replaced with a curve on operating point ($Q$) as can be seen in Fig.2. In Fig.2.b, the element is assumed as linear around the point-$Q$. It can be expressed around this point as below:

$$x(t) = X_0 + x_a(t) \qquad (1)$$

Where $X_0$ is a direct component and independent from $t$. Whereas, $x_a(t)$ is a variable component and it depends on $t$. While variable components are expressed as Eq.(2), these are called small-signal.

$$x_a(t) = |x(t) - X_0| \ll X_0 \qquad (2)$$

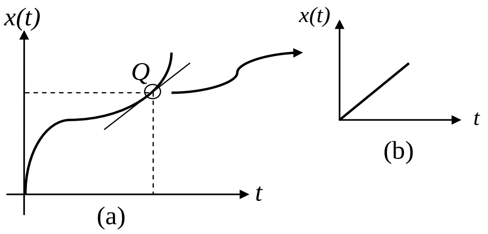


Fig.2. Linearization on operating point ($Q$)

## III. Large-Signal Analysis

The equations which are obtained for large-signal analysis of the circuits including nonlinear elements are nonlinear algebraic equations. The method used for the solution of these equations is Newton-Raphson algorithm.

Nonlinear circuits require solution of the nonlinear algebraic equation systems and the general structure of these equations is as in Eq.(3). The aim, here, is to find the root of the function.

$$f(x) = 0 \qquad (3)$$

The solution expression of the nonlinear equations, Eq.(3), with Newton-Raphson iterative method is given as below:

$$x^{j+1} = x^j - \frac{f(x^j)}{f'(x^j)} \qquad (4)$$

Nonlinear multivariate algebraic equation system is given by Eq.(5):

$$F(X) = 0 \Rightarrow \begin{matrix} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_1(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0 \end{matrix} \qquad (5)$$

Newton-Raphson formula is given as the following for the equation system given by Eq.(5):

$$X^{j+1} = X^j - \frac{F(X^j)}{F'(X^j)} \qquad (6)$$

$$X^{j+1} = X^j - J_F^{-1}(X^j)F(X^j) \qquad (7)$$

Where,

$$J_F(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad (8)$$

In Eq.(7), the expression $F'(X^j) = J_F(X^j)$ is Jacobian matrix of the function $F(X)$.

Fig.3 shows Newton-Raphson algorithm concerning the solution method.

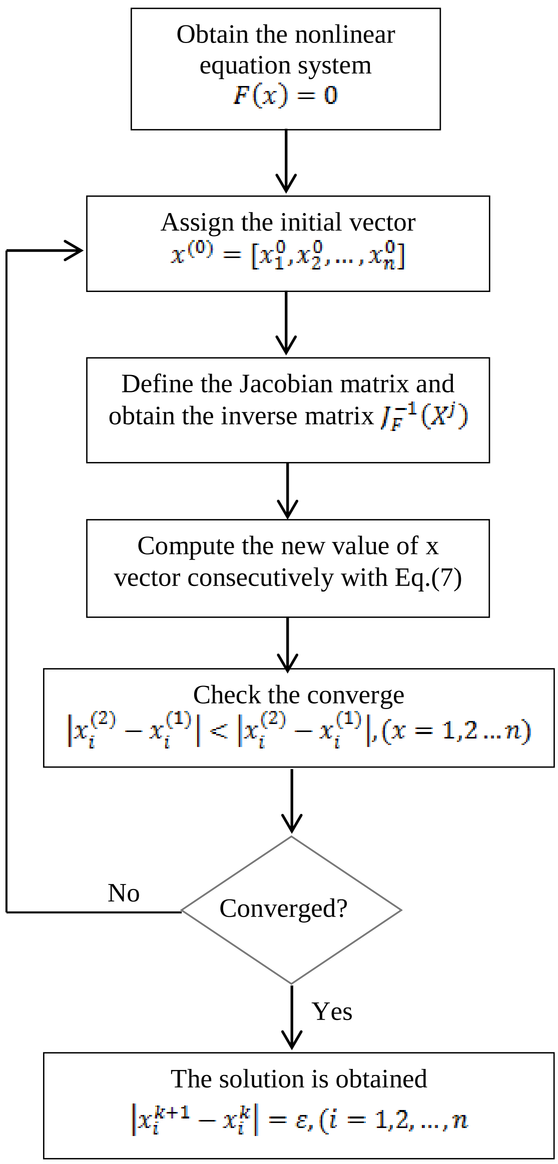


Fig.3. Newton-Raphson Algorithm

## IV. Nonlinear Three-Terminal Switching Elements

The most important three-terminal elements we encounter in real world are transistors (BJT, JFET, MOSFET… etc.). Their global symbol and terminal-graph are illustrated in Fig.4.

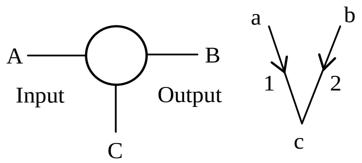

Fig.4. The symbol and terminal-graph of a three-terminal element

The terminal equations of a three-terminal element shown in Fig.4 can be one of four different forms as below.

$$(a)\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} f_1(v_1,v_2) \\ f_2(v_1,v_2) \end{bmatrix} \quad (b)\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} f_1(i_1,i_2) \\ f_2(i_1,i_2) \end{bmatrix} \quad (9)$$

$$(a)\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} f_1(i_1,v_2) \\ f_2(i_1,v_2) \end{bmatrix} \quad (b)\begin{bmatrix} i_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} f_1(v_1,i_2) \\ f_2(v_1,i_2) \end{bmatrix} \quad (10)$$

General constructions of Eq.(9) and Eq.(10) are given by:

$$u = f_1(x,y) \qquad (11)$$

$$v = f_2(x,y) \qquad (12)$$

Where, each one of the expressions represents a bivariate function and therefore, these correspond a surface in 3-dimensional space.

If a three-terminal element operates with small-signal variations around the operating point $(X_0, Y_0)$, this element can be linearized as in Eq.(13) and Eq.(14).

$$U_0 = f_1(X_0, Y_0) \qquad (13)$$

$$V_0 = f_2(X_0, Y_0) \qquad (14)$$

For linearization, Taylor series expansion is used:

$$u = f_1(X_0,Y_0) + \frac{\partial f_1}{\partial x}\Big|_Q (x - X_0) + \frac{\partial f_1}{\partial y}\Big|_Q (y - Y_0) + \frac{1}{2!}\frac{\partial^2 f_1}{\partial x^2}\Big|_Q (x - X_0)^2 + \cdots \qquad (15)$$

$$v = f_2(X_0,Y_0) + \frac{\partial f_2}{\partial x}\Big|_Q (x - X_0) + \frac{\partial f_2}{\partial y}\Big|_Q (y - Y_0) + \frac{1}{2!}\frac{\partial^2 f_2}{\partial x^2}\Big|_Q (x - X_0)^2 + \cdots \qquad (16)$$

where, high degree terms can be negligible because of $|(x - X_0)| \le X_0$ and $|(y - Y_0)| \le Y_0$ in operating point $Q = Q(X_0, Y_0)$, and so Eq.(17) and Eq.(18) are obtained.

$$u = U_0 + \frac{\partial f_1}{\partial x}\Big|_Q (x - X_0) + \frac{\partial f_1}{\partial y}\Big|_Q (y - Y_0) \qquad (17)$$

$$v = V_0 + \frac{\partial f_2}{\partial x}\Big|_Q (x - X_0) + \frac{\partial f_2}{\partial y}\Big|_Q (y - Y_0) \qquad (18)$$

Let's substitute the following equations into Eq.(17) and Eq.(18):

$$u - U_0 = u' \text{ and } x - X_0 = x'$$

$$v - V_0 = v' \text{ and } y - Y_0 = y'$$

After that, Eq.(17) and Eq.(18) are in the following form.

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial x} \end{bmatrix}_Q \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad (19)$$

where, the matrix which is multiplied with the vector $[x' \quad y']^T$ is Jacobian matrix ( $J_Q$). Eq.(19) can be written also by Eq.(20).

$$\begin{bmatrix} u - U_0 \\ v - V_0 \end{bmatrix} = J_Q \begin{bmatrix} x - X_0 \\ y - Y_0 \end{bmatrix} \qquad (20)$$

The geometrical meaning of linearization is that the surface equation is replaced with a plane equation which is tangent into the surface in operating point, *Q*.

## V. APPLICATION

Consider the MOS transistor circuit shown in Fig.6. Element values are $R1 = 2k\Omega$, $R2 = 400\Omega$, $E1 = 8V$, $E2 = 12V$, $e(t) = 0.01sint$, also $\beta = 0.64mA/V^2$, $V_{th} = -5V$. The terminal equations of MOS transistor [2]:

$$i_1 = 0 \qquad (21)$$

$$i_2 = \beta\left[(v_1 - v_{th})v_2 - \frac{1}{2}{v_2}^2\right] \qquad (22)$$

The nonlinear expression given by Eq.(22) is shown as 3-dimensional in Fig.5.

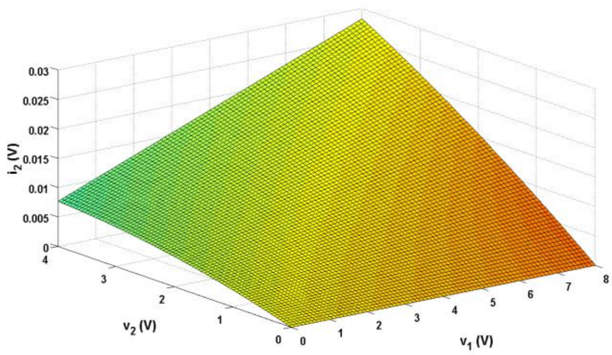

Fig.5. 3-D representation of Eq.(22)

At first, let's find the operating point values in order to obtain the large-signal analysis. Therefore, the circuit shown in Fig.7 is obtained by short-circuiting the alternating source in the main circuit (Fig.6). Mesh equations are as follows:

$$\rightarrow -E_1 + I_a R_1 + V_1 = 0 \qquad (23)$$

$$\rightarrow -E_2 + I_b R_2 + V_2 = 0 \qquad (24)$$

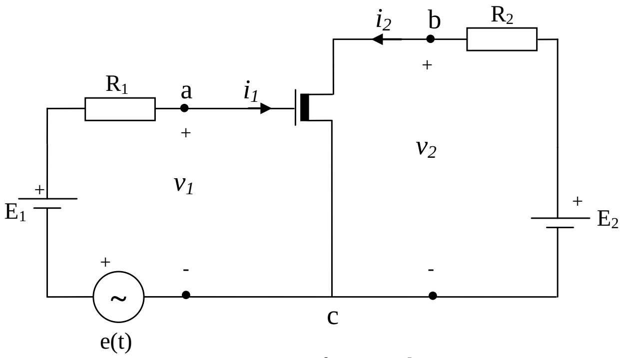


Fig.6. Circuit for example

The variables of the method are $I_a = i_1$ and $I_b = i_2$ mesh currents. The expressions of Eq.(21) and Eq.(22) are used for $i_1$ and $i_2$, then let's rearrange Eq.(23) and Eq.(24):

$$\rightarrow g_1(V_1, V_2) = V_1 - 8 = 0 \quad (25)$$

$$g_2(V_1, V_2) = V_2 - 12 + 0.4\beta\left[(v_1 - v_{th})v_2 - \frac{1}{2}{v_2}^2\right] = 0 \quad (26)$$

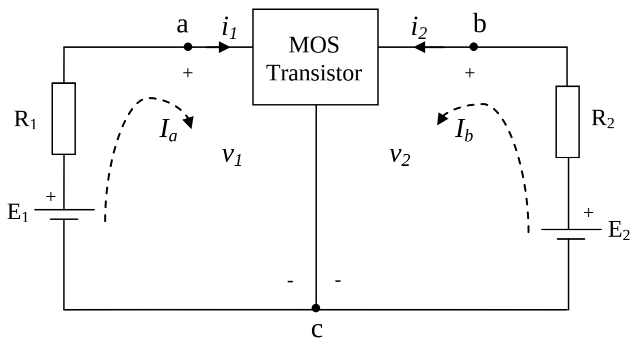


Fig.7. The circuit in operating point ($Q$)

Newton-Raphson iterative formula given by Eq.(7) is applied this equation system as follows:

$$\begin{bmatrix} {V_1}^{j+1} \\ {V_2}^{j+1} \end{bmatrix} = \begin{bmatrix} {V_1}^{j} \\ {V_2}^{j} \end{bmatrix} - \frac{1}{1 + 0.4\beta[({V_1}^j - V_{th}) - {V_2}^j]} \quad (27)$$

The solutions are obtained for initial values given as ${V_1}^{(0)} = 8$, ${V_2}^{(0)} = 1$, as follows.

$$V_{1Q} = 8V$$

$$V_{2Q} = 3.05V$$

Now, let's start the small-signal analysis. It can be easily seen that the terminal equations given by Eq.(21) and Eq.(22) are in the form of Eq.(9.a). After the structure of equation is determined, the Jacobian matrix in Eq.(19) is expressed and the linearized equivalent circuit of three-terminal element is obtained as in Fig.8.

$$\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} I_{10} \\ I_{20} \end{bmatrix} + J_Q \begin{bmatrix} v_1(t) - V_{10} \\ v_2(t) - V_{20} \end{bmatrix} \quad (28)$$

$$J_Q = \begin{bmatrix} \frac{\partial i_1}{\partial v_1} & \frac{\partial i_1}{\partial v_2} \\ \frac{\partial i_2}{\partial v_1} & \frac{\partial i_2}{\partial v_2} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

$$J_Q = \begin{bmatrix} 0 & 0 \\ \beta v_2 & \beta[(v_1 - v_{th}) - v_2] \end{bmatrix} \quad (29)$$

By substituting Eq.(29) into Eq.(28), Eq.(30) is obtained as follows.

$$\begin{bmatrix} {i_1}' \\ {i_2}' \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} {v_1}' \\ {v_2}' \end{bmatrix} \quad (30)$$

Fig.8 is the linearized equivalent circuit of three-terminal element. It is created by using the expression of Eq.(30).

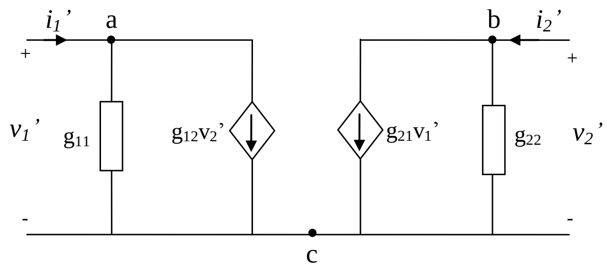


Fig.8. The linearized equivalent circuit of MOS transistor

To achieve the small-signal analysis, direct sources are short-circuited in the main circuit in Fig.6 and the linearized equivalent circuit in Fig.8 is taken into account. As a result, the complete equivalent circuit is obtained as in Fig.9. The values of resistors in the equivalent circuit are determined by substituting the values of $V_{1Q}$ and $V_{2Q}$ into Eq.(29).

$$J_Q = \begin{bmatrix} \frac{\partial i_1}{\partial v_1} & \frac{\partial i_1}{\partial v_2} \\ \frac{\partial i_2}{\partial v_1} & \frac{\partial i_2}{\partial v_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta v_2 & \beta[(v_1 - v_{th}) - v_2] \end{bmatrix}_Q$$

$$J_Q = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.6 \end{bmatrix} \quad (31)$$

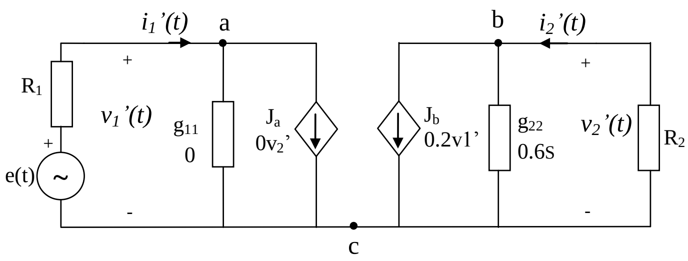


Fig.9. The equivalent circuit obtained for small-signal analysis

The values relating to small-signal analysis are obtained from Fig.9. Here, $v_a(t) = {v_1}'(t)$ and $v_b(t) = {v_2}'(t)$.

$$\rightarrow v_1'(t) - e(t) + i_1'(t)R_1 = 0 \qquad (32)$$

$$\rightarrow v_2'(t) + i_2'(t)R_2 = 0 \qquad (33)$$

Nodal equations (for nodal a and b) are as follows:

$$\rightarrow i_1'(t) - 2.5v_1'(t) - J_a = 0 \qquad (34)$$

$$\rightarrow i_2'(t) - 1v_2'(t) - J_b = 0 \qquad (35)$$

After solving the system equations from Eq.(32) to Eq.(35), nodal voltages are obtained as follows:

$$\rightarrow v_1'(t) = 0.01sint$$

$$\rightarrow v_2'(t) = -0.0022sint$$

Finally, the complete solutions are given by following expressions.

$$\rightarrow v_1(t) = V_{1Q} + v_1'(t) = 8 + 0.01sint$$

$$\rightarrow v_2(t) = V_{2Q} + \square_2'(t) = 3.05 - 0.0022sint$$

## VI. CONCLUSIONS

In this paper, the large-signal and small-signal analysis of nonlinear three-terminal switching elements we encounter frequently in practice are carried out. To obtain and analyze nonlinear equation systems require intensive mathematical operations. In this study, for large-signal analysis, a frequently used method, Newton-Raphson method is used. Also, an application circuit containing MOS transistor as a three-terminal switching element is included in the study. The voltages in operating point, *Q*, are obtained with the large-signal analysis relating to this element. Also, the effect of alternating source to the circuit is studied by small-signal analysis. The method can be easily applied to the circuits containing nonlinear resistor or nonlinear multi-terminal other elements. The main difficulty in the solution process is to obtain nonlinear equation system and deal with non-algebraic operations.

## REFERENCES


[1] D. Maksimovic, A. M. Stankovic, V. J. Thottuvelil, G. C. Verghese, "Modeling and Simulation of Power Electronic Conventers", *IEEE Proc.*, vol. 89, no. 6, pp. 898-912, June 2001.

[2] L. O. Chua, C. A. Desoer, E. S. Kuh, "Linear and Nonlinear Circuits", McGraw-Hill Press, 1987.

[3] L. O. Chua, "Device Modeling Via Basic Nonlinear Circuit Elements", *IEEE Transactions on Circuits and Systems*, vol. cas-27, no. 11, pp. 1014-1044, Nov. 1980.

[4] A. Ushida, T. Adachi, L. O. Chua, "Steady-State Analysis of Nonlinear Circuits Based on Hybrid Methods", *IEEE Transactions on Circuits and Systems*,- I: Fundamental Theory and Applications, vol. 39, no. 8, pp. 649-661, Aug. 1992.

[5] A. Opal, "Sampled Data Simulation of Linear and Nonlinear Circuits", *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, vol. 15, no. 3, pp. 295-307, March 1996.

[6] L. O. Chua, "Nonlinear Circuits", *IEEE Transactions on Circuits and Systems*, vol. cas-31, no. 1, pp. 69-87, Jan. 1984.

[7] F. N .Najm, "Circuit Simulation", John Wiley & Sons Press, 2010.

[8] D. Hente and R. H. Jansen, "Frequency-Domain Continuation Method for the Analysis and Stability Investigation of Nonlinear Microwave Circuits", *IEEE Proc.*, vol. 133, no. 5, pp. 351-362, Oct. 1986.

[9] A. B. Yildiz, "Electric Circuits 1, Theory and outline problems", Volga press, 2014.

[10] L. Nagel, R. Rohrer, "Computer Analysis of Nonlinear Circuits, Excluding Radiation (CANCER)", *IEEE Journal of Solid-State Circuits.*, vol. sc-6, no. 4, pp. 166-182, Aug. 1971.

[11] E. V. Damme, J. Verspecht, F. Verbeyst, M. V. Bossche, "Large-Signal Network Analysis – a measurement concept to characterize nonlinear devices and systems", Agilent Technologies, 2002.

[12] I. Dobson, "Stability of Ideal Thyristor and Diode Switching Circuits", *IEEE Transactions on Circuits and Systems*, vol. 42, no. 9, pp. 517-529, September 1995.

[13] C. Turchetti, M. Conti, "A General Approach to Nonlinear Synthesis with MOS Analog Circuits", *IEEE Transactions on Circuits and Systems*, vol. 40, no. 9, pp. 608-612, September 1993.

[14] R. A. Minasian, "Power MOSFET Dynamic Large-Signal Model", *IEE Proc.*, vol. 130, pt. I, no. 2, pp. 73-79, April 1983.

[15] M. Fantao, et al., "Nonlinear electrical properties of Si three-terminal junction devices", *AIP Journal & Magazines*, vol.97, issue 24, pp, 242106-142106-3, 2010.

[16] D. Wallin, et al., "Nonlinear electrical properties of three-terminal junctions", *AIP Journal & Magazines*, vol.89, issue 9, pp, 92124-92124-3, 2006.